THE

MATHEMATICAL GAZETTE

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STUDY AND RESEARCH IN MATHEMATICS.

BY H. DAVENPORT.

(*Presidential address to the London Branch of the Mathematical Association,* 15 *October*, 1949.)

I PROPOSE to put before you today a few somewhat disconnected thoughts and reflections that have occurred to me as a result of my experiences as a student, as a researcher, and as a teacher. I cannot say that I think they are of much value ; some of them are obvious, and others are merely expressions of opinion. But it is good for us to stand back occasionally from our subject and look at it as a form of human activity, and an address like this provides a convenient opportunity for doing so.

Study and research have much in common. In both of these activities, an individual is striving to acquire knowledge which is new to him. He is engaged in discovery. When studying, he has guides in the form of teachers and books to help him, and even when he is not actually making use of them he has the comforting knowledge that they are at hand if he needs them. In research, he has to rely almost entirely upon his own efforts. Students vary a great deal in the extent to which they depend on their teachers and books, but it is a familiar fact that the more a student can discover for himself, the greater the mastery he achieves over the knowledge he has won. In other words, the most effective study is that which most closely resembles research.

There is one thing upon which we probably all agree, and that is that mathematics is a difficult subject! How much effort one puts into studying an infinitesimal fraction of the subject, and how even so one only acquires an imperfect comprehension of it and is easily baffled by simple questions! How often one deceives oneself over many years into thinking one thoroughly understands something, and then discovers by chance that one has never appreciated a crucial point! Such reflections as these must, I think, always preserve us from any feeling of complacency or arrogance.

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One obvious reason for the difficulty of mathematics is the enormous extent of the subject and its constant growth. When a student leaves school and proceeds to the University, he has learnt a fair proportion of the mathematics that was discovered (or created) up to about 1700 or 1750. This incorporates the life-work of many geniuses; and although what was found by one man may later have been simplified by others, yet nothing has been rejected and the body of knowledge has constantly grown. The modern development of mathematics has been so rapid and has proceeded in so many different directions that it is impossible for an undergraduate, in three years of study, to become acquainted with more than a minute fraction of it. Certainly ours is a subject in which the shortness of human life makes itself painfully felt, especially since after the age of 30 or so it becomes difficult to master new ideas to such an extent that they become part of oneself.

There is another way of looking at the difficulty of mathematics which has suggested itself to me. Is it perhaps not so much that mathematics is difficult as that we human beings are stupid? It often seems to me that the human mind does not very well adapt itself to mathematical or logical thought. When trying to solve a particular problem, I sometimes feel as if I were in a labyrinth: that my mind persists in laboriously exploring one avenue after another and coming to a dead end, often to the same dead end more than once. Frequently it turns out in the end, when the problem is solved, that there was one "right" path, and that there were perfectly good reasons for choosing that path at the beginning, reasons which one simply did not see, though they ought to have been plain enough. Such an experience makes me think rather of my own stupidity than of the inherent difficulty of the subject.

When I look back upon the important discoveries that have been made in the theory of numbers during the last twenty years, I find that there are several about which one feels surprised that they were not made earlier. Some of them relate to problems which were deeply studied, without success, by mathematicians of the greatest ability; and yet when one analyses the solution it turns out to involve nothing that was necessarily beyond their powers or entirely outside their range. A topical example is the elementary proof of the Prime Number Theorem, the theorem which asserts that $\pi(n)$, the number of primes among the numbers $1, 2, 3, \ldots, n$, satisfies the limit-relation $\pi(n) \log n/n \to 1$ as $n \to \infty$. A proof was sought for by many great mathematicians throughout the nineteenth century, and the search for an "elementary" proof* was not abandoned after an analytical proof had been found by Hadamard and de la Vallée Poussin in 1896. The first elementary proof was finally obtained by Selberg and Erdös in 1948. The highest credit is due to them for their remarkable achievement. The proof is a fine piece of work, and by no means easy; yet on studying it, one feels a little surprised that it should have escaped discovery so long, in view of the strenuous efforts that were made. Both Selberg and Erdös have other great achievements to their credit which are probably deeper and of more profound significance.

In contemplating the great mathematical achievements both of the past and of the present, one is overcome by admiration for the qualities of mind that produced them. This feeling is strengthened in the case of many past achievements when one recalls the absence, at that time, of techniques which are now familiar. Certainly they were exceptional minds, and I am inclined to think that their discoveries represent the results of exceptional efforts, even relative to those minds. It may still be true that the human mind in general is ill adapted for mathematical thought.

* An elementary proof, in this connection, means one which uses only the integers. In particular, no appeal to the theory of functions is allowed.

The mention of "exceptional minds" recalls the old problem of whether there is such a thing as a specific and inborn talent for mathematics. Certainly appearances strongly suggest that there is: one may see two children subjected to the same education and with similar general intelligences (as far as that can be judged), and the mathematical ability of one may be so much greater than that of the other as to be beyond all comparison. Yet appearances may be deceptive. Much may be due to the child's interest being aroused by mathematics at a suitable moment, and this interest persisting and deepening.

There is nothing more curious than the varying potentialities of the human mind according as *interest* is or is not aroused. In itself, this is a truism, but mathematics is one of those subjects in which the phenomenon shows itself very clearly. Music is another. There *are* subjects in which one can progress quite well up to a fairly advanced stage of study without having a passionate interest in the details of the subject. There must be *some* motive, of course, but it may be one of duty or emulation or of an ulterior benefit which is expected to accrue. It seems to me that such motives do not carry one very far in mathematics, though they may serve to carry one to a point at which interest in the subject for its own sake is aroused.

It may be, therefore, that the tremendous variation in ability shown by students of mathematics is not due to the presence or absence of an inborn talent, but to whether their interest has been aroused. What it is that excites interest seems obscure, but I should think that the process depends on causes which are so varied and unpredictable as to be almost equivalent to chance. I can imagine, for example, a child developing an interest in French because the map of France in his atlas happened to be coloured in his favourite colour.

In mathematics, there is no lack of unexpected and striking facts to arouse interest. In my own case, I can remember the excitement that I felt when I first saw the infinite series and products for the sine and cosine, and again when I was shown the identity

$$\frac{1}{1-x} = (1+x)(1+x^2)(1+x^4) \ldots .$$

Indeed, now I look at this identity again, it still seems to be quite an interesting one. It is proved, of course, by writing the factors on the right as $(1-x^2)/(1-x)$, and so on, and then cancelling. Having proved the identity, one can expand the left hand side as $1+x+x^2+\ldots$, and on comparing this with the expansion of the right hand side one gets a proof that every natural number is representable uniquely as a sum of distinct powers of 2 (including $2^0=1$). We have here a simple illustration of the way in which the introduction of a continuous variable can lead one, by a few strokes of the pen, to a theorem about the natural numbers.

I remember also being intrigued by the various kinds of proof, and puzzled by the existence of proofs which, although logically perfect, could still leave one mentally unsatisfied. But it may well be that there were other causes, not known to me, for my becoming interested in mathematics.

Certainly it is our supreme duty, as teachers, to do all we can to arouse the interest of our students, and do nothing which may damp their enthusiasm. I know how trite this sounds, but yet it is something of which we cannot remind ourselves too often. I think much can be done by stressing the different ways of looking at each topic, and the curious connections between various topics, connections often not remarked upon in textbooks. We should encourage our students to find things out for themselves, and acquire the outlook of a discoverer and not that of a spectator. Perhaps it is a good thing to let them realise that there are many things which we ourselves do not fully understand.

One thing that we, as teachers, have to beware of is the natural but insidious theory that in teaching a particular part of mathematics there is a "simplest" method, and that we ought to find it and follow it. This is an attitude which one has constantly to struggle against. In the first place, what seems simple and natural to one student may not appear so to another; and in the second place, once a teacher has chosen one particular method, his teaching inevitably loses the spontaneity which is essential if he is to arouse and retain the interest of students.

I will now pass on to another topic, and say a little about a division running throughout mathematics, the existence of which is not always realised; that is, the division into *discrete* and *continuous* mathematics. By discrete mathematics one means those subjects into which the notion of continuous variation does not enter, or need not enter. Such subjects are algebra, including the theory of numbers and the theory of groups, and pure geometry. On the other hand, analysis, analytical geometry, and conventional applied mathematics are all dominated by the notion of continuous variation. The distinction is one that has only become plain in modern times, and so has hardly yet been reflected in educational methods. In elementary algebra, we assume tacitly that the symbols represent real numbers, or "quantities", capable of continuous variation, though in fact such an assumption is quite foreign to the nature of algebra. Algebra (in the modern view) is concerned with operations, primarily the four operations of arithmetic, applied to systems of elements whose nature is immaterial. It is only in recent times that discrete mathematics has freed itself from continuous mathematics, and since doing so has developed out of all recognition.

I think that many of us find continuous mathematics comes more easily and naturally to us than discrete mathematics. Some allowance must be made for the bias of mind produced by our education, which reflects the historical development of mathematics. But there may be a deeper reason. The notions of *time* and *movement* and *variation* are certainly very deeply rooted in our minds, and it seems to me that one may well feel handicapped in reasoning about problems where no analogy with these notions is possible. I suppose that, from a strictly logical point of view, the notion of time has no place in mathematics: such phrases as "$f(x)$ increases steadily with x" are used only as concessions to human weakness. But from the human point of view, the whole of continuous mathematics arises out of this analogy with time. The lawyers have a phrase "time is of the essence", and I think this applies, in a different sense, to the operation of the human mind.

Finally, I would like to touch briefly upon another question, that of the process of discovery or invention, in mathematics. This has been discussed by two great French mathematicians, Poincaré in *Science et Méthode* and Hadamard in his reçent book *The Psychology of Invention in the Mathematical Field*. Hadamard, in particular, has gone into the question very fully, but the most interesting part of his book concerns his personal experiences, which are very similar to those of Poincaré. Both of them describe the process of discovery as comprising four stages. The first is that of exploration and preparation, of finding out where the essential difficulty lies and removing any misconceptions. The second stage is that of incubation, during which one fails to make any apparent progress, but nevertheless the situation revealed by the stage of preparation sinks more deeply into one's mind, and is worked on both consciously and (as Poincaré and Hadamard believe) subconsciously. The third stage is that of revelation. At some quite unexpected moment an idea flashes across the mind which provides the answer. In an experience described by Poincaré, the idea came to him as he was getting on a bus while taking part in a geological excursion. Sometimes the inspiration takes the

form of an immediate illumination of the whole situation, sometimes it consists of a clue to the solution of the problem. Finally, there is the stage of verification in which the new work is consolidated.

Both Poincaré and Hadamard have stressed the part played by the subconscious, and this is certainly a strange phenomenon, well worthy of attention. It is so striking that there is a danger of its importance being exaggerated. What is quite certain is that the subconscious mind comes into action only after a great amount of conscious work has been done, and much effort has been expended on the problem. What the subconscious mind accomplishes may be only a slight " prolongation " of a train of thought which was already going on in the conscious mind. I cannot say that I have ever experienced this subconscious working of the mind myself, at any rate to a sufficient degree to be certain of it.

I have used the words discovery *or* invention. Which word best describes original work in mathematics? The distinction in common usage is plain enough : " discovery " relates to an object or phenomenon or law which was already in existence but was not perceived, " invention " calls into existence something new. But what is " existence "? Do mathematical systems and theorems exist before they have been formulated by human beings, or not? A clear-cut answer seems impossible, and original work in mathematics seems to combine discovery and invention.

Which word one prefers may depend on the view one takes of the nature of mathematics. Are the things we talk about (numbers, functions, spaces, etc.) real or hypothetical? Neither answer seems adequate by itself. " Reality " is a vague and deceptive notion, but on the other hand the study of hypothetical systems of elements satisfying certain axioms may be devoid of significance if we have no assurance that the system is self-consistent. On the " real " view, mathematical results are discoveries, on the "hypothetical " view they are inventions. The truth seems to lie somewhere in between.

The hypothetical or axiomatic view of mathematics has been widely adopted in the last fifty years. According to this view, we lay down certain axioms, without further specification of the objects to which they are to apply, and develop their consequences by applying the laws of logic. Bertrand Russell probably had this view in mind when he produced his epigram that " in mathematics we do not know what we are talking about, and we do not care whether what we say about it is true or not ". We do not know what we are talking about because the things we are talking about are hypothetical entities satisfying axioms, and we do not know whether what we say is true because " truth " has been replaced by " a logical consequence of the axioms ". This seems to me to be fallacious. In fact we all have a strong conviction that we sometimes *do* know what we are talking about in mathematics, and that some of the propositions we reach are true in a stronger sense than that of logical deduction. It seems to me that axioms and the laws of logic provide a mechanism for mathematics, but do not tell us all about it. An analogy which comes naturally to mind is that of music. Suppose one points to a piano and says : here is an instrument with keys ; *music* is the noise which results when these are struck in accordance with certain rules. In a sense this is true, but it conveys no idea of the nature of music. What constitutes music we are unable to say, and perhaps it is not surprising that it is equally difficult to say what is mathematics.

H. D.

DEMONSTRATION APPARATUS IN THE TEACHING OF APPLIED MATHEMATICS.

By J. C. Jaeger.

Several authors have recently discussed in the *Gazette* the place of experiment in the teaching of mathematics. Although it is true that most students can visualise the behaviour of a mechanical system quite well from a proper description or a good drawing, I find that all students, and particularly engineers, seem to be greatly stimulated by an occasional demonstration related to their work.

For some time I have been trying to construct courses in applied mathematics in which all the examples studied have obvious practical applications. This need not involve any lowering of the standard of the mathematical work, but it does involve some changes in the type of problem discussed.

Students have to be taught to do three things : (i) to express the working of a practical mechanical system in mathematical terms, say as a set of differential equations ; (ii) to solve these equations ; and (iii) to express their solutions in a workmanlike form from which numerical calculations can be made or physical conclusions drawn. The development of skill in the last part of the programme, which consists mainly of algebraic or trigonometric manipulation, is of the greatest importance. If, in addition, the eye can be caught by a working model and some of the general features of a solution quickly demonstrated by it, I believe the average student benefits greatly.

One of the best systems, both for practice in mathematical manipulation and for demonstration, is the vibration of a set of coupled masses. By a curious convention, the teacher of applied mathematics has usually to confine himself to mechanical problems. This is, from the present point of view, no disadvantage, as they are easy to demonstrate and at the same time can be given added interest by indicating the analogous systems for linear motion, rotational motion, and electrical and acoustical circuits.

The apparatus shown in the figures was constructed to demonstrate a wide variety of such problems. It is of careful and robust construction, and I think this of importance for two reasons : firstly, because in a light structure unwanted effects may appear, and secondly because a nice piece of workmanship will itself interest the mechanically-minded. Construction of this sort implies that Applied Mathematics must have a small workshop, and indeed it is ridiculous that mathematicians are expected to teach the backbone of engineering and physics with no apparatus beyond ruler and compasses.

The most useful arrangement is shown in Fig. 1. Three discs are mounted on ball bearings and connected by elastic shafts ; at the lowest bearing there is a crank which is oscillated very nearly in simple harmonic motion by a variable speed motor. Either one, two, or three discs may be used, and any disc can be clamped. The discs are 6″ diameter and ½″ thick (two can be used together to double the moment of inertia). Welding electrodes are used for the shafts since a wide range of diameters is available in them. Normally no damping is used and the oscillations have quite large amplitudes. Marks on the discs allow the movements of the discs relative to scales to be watched, and a pointer indicates the position of the crank.

Now for the systems which can be demonstrated, the study of each of which has a useful amount of mathematics in it.

(i) *One disc and shaft.* The variation with frequency of the amplitude and phase of the oscillations of the disc. A crude resonance curve can be drawn if desired, and rough calculations of the stiffness of the shaft and the damping coefficient made.

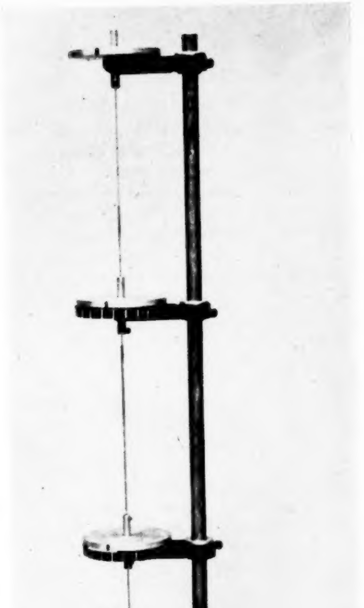


Fig. 1.

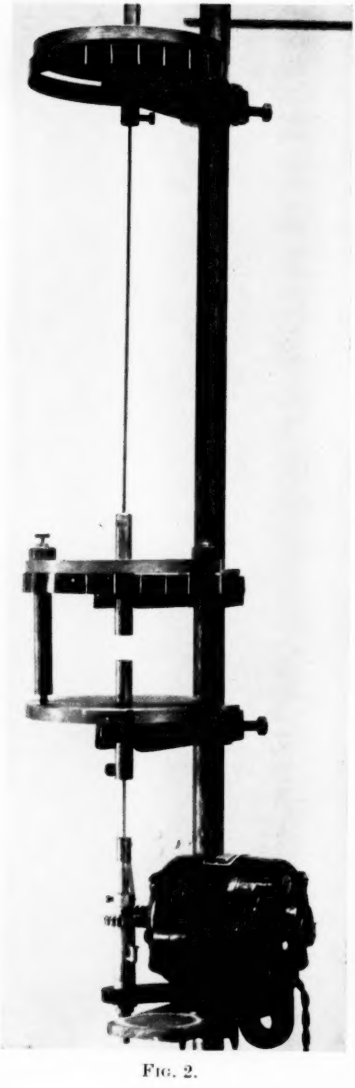


FIG. 2.

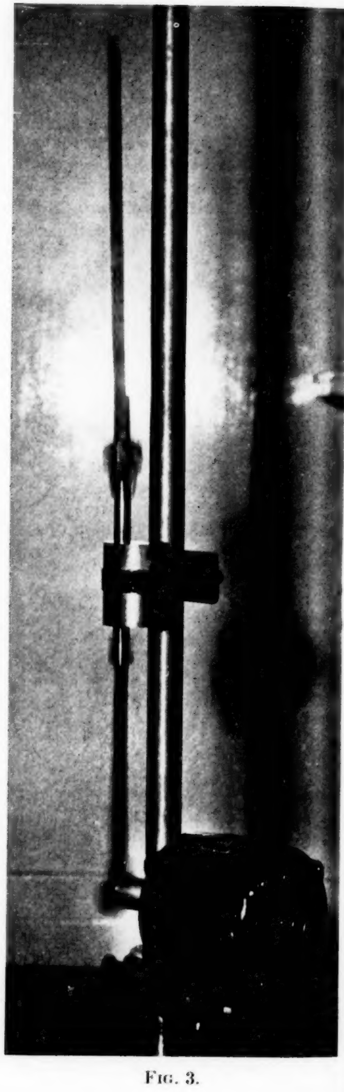


FIG. 3.

(ii) *Two discs.* The two natural frequencies and normal modes. The effect of doubling the moment of inertia of the upper disc. The effect of clamping the upper disc.

(iii) *Three discs.* The three natural frequencies and normal modes. Clamping the upper disc gives another set of examples with two normal modes.

(iv) *Transients.* The effect of the initial conditions on the type of vibration. Building up of oscillations when motion of the crank is started suddenly.

(v) *Combined harmonic motions.* An epicyclic motion gives the crank combined simple harmonic oscillations with frequencies in the ratio 4 : 1. The two peaks in the resonance curve appear.

(vi) *Higher harmonics produced with a short connecting rod.* The first of these shows up well if the length of the connecting rod is about twice that of the crank.

(vii) *Shaft whirling.* An example of an " eigenvalue " problem. The motor is turned round to drive a shaft directly : a disc half-way between the bearings whirls spectacularly.

(viii) *Relaxation oscillations.* In Fig. 2 the apparatus is arranged to demonstrate the " stick-slip " vibrations which occur when the coefficient of dynamic friction is less than the coefficient of static friction. The motor drives the lowest disc very slowly as a turntable. The two rods projecting from the second disc carry hard rubber sliders which press on the turntable ; the pressure is adjusted by screws at the top. This disc is connected by an elastic shaft to the top disc which is clamped. The action is as follows : the second disc moves with the turntable until the torque due to twisting of the shaft becomes equal to that due to static friction at the contacts ; it then slips backwards, the motion being resisted by the lesser force of dynamic friction until relative rest, when the process repeats.

(ix) *The pendulum with vertical motion of its support.* The point of support of the pendulum is driven from the motor by a slider-crank mechanism. The stabilising of an inverted pendulum is shown in Fig. 3. The production of instability in a hanging pendulum also shows up very well. The theory is, of course, more difficult than that of the other experiments, but it is desirable for students to understand the principles involved and to be able to use the stability chart for Mathieu's equation.

Clearly many more applications of this apparatus could be catalogued. These experiments take a very short time to show—in fact, little longer than drawing a diagram and explaining it.

Finally, I must gratefully acknowledge the assistance of Mr. J. D. Clarke, one of my students, who has built this and much other similar apparatus for me.

J. C. J.

GLEANINGS FAR AND NEAR.

1642. In the communication of mind with mind language serves two functions. It conveys thought and it conveys feeling. Words primarily are used to express intellectual concepts, and the best words are the ones which utter the conceptions with clearness and precision. But words are like worlds ; they are dead unless they have an atmosphere. Pure thought may have currency among celestial beings and mathematicians, but among ordinary mortals every thought is charged with an appropriate emotion.—C. A. Dinsmore, *The English Bible as Literature,* quoted by W. R. Bowie in *The Use of the New Testament in Worship.* [Per Professor N. Anning.]

THE TEACHING OF MECHANICS.*

THE ELEMENTARY COURSE.

Mr. K. S. Snell (Harrow) :

The teaching of Mechanics is a very wide subject for a short discussion. I have for a long time considered that the elements of Mechanics provide very suitable topics to enliven a course of mathematics, and hence I am devoting most of my introduction to the elementary course, leaving Dr. Baggott to say more about the later school and hence more specialist course. What I have to say must concern mostly boys and girls in secondary grammar schools. Those in technical schools clearly all do a course in Mechanics already, as well as other mathematics. In secondary modern schools isolated applications to Mechanics would occur, especially in the graphical work, but clearly Mechanics would not occur as a subject. But on reading that excellent interim report just issued by the Association I saw how certain Mechanics applications would occur—especially in the " Railway " project.

There is one other point in introduction. Many boys and girls will now, under the new examination scheme, continue mathematics to the age of 16, when they might previously have stopped earlier. What are we going to teach them? Clearly it must not be a repetition of a limited examination syllabus. I put forward as a suggestion the claims of mechanics.

I shall suggest four main points, in the hope that they may give something definite as a lead for the ensuing discussion.

(1) Mechanics is the easiest and best section of " additional mathematics " for most students. The beginnings of the subject deal with concepts already familiar to them—the lever, speed and acceleration, effect of force in changing velocity. In all of these the language is familiar, and there is a field ready for developing by means of elementary mathematics. Inverse proportion leads to the principle of moments ; graphical representation can expand knowledge of distance, velocity, acceleration, with no restriction to constant acceleration ; knowledge of trigonometrical ratios can be extended to the effect of a force in a given direction or to compounding components of velocity ; ratio can be applied to machines of various kinds. There are no new complicated formulae to learn, but the essential parts of elementary mathematics —solution of equations, use of tables, knowledge of simple configurations— are all used.

Compare with this the advance in other mathematical directions. More deductive geometry is excellent for the future mathematician, but meaningless to many. Further algebra—progressions, indices, etc.—can be made to appeal, but again tend to be of more value to the mathematician. Trigonometry has many applications, but once the three ratios have been learned and possibly the sine and cosine rules, further advance to compound angles, etc., is not practical but theoretical, and further applications, such as those to Mechanics, are of the greatest value. The elements of calculus are now much in favour, especially since the Jeffery report on examinations. The elements are normally approached geometrically, *via* gradients, and are difficult for the average student. I maintain that it is easier to give a graphical representation of speed and acceleration and to lead on to methods of calculating them, and thus incidentally to the beginnings of differential calculus. Again, the easiest approach to the calculation of area by integration is through graphical kinematics. Further, the ideas of elementary calculus are not

* A discussion at the Annual Meeting of the Mathematical Association, 3rd January, 1950.

familiar to the student—they provide good extensions of elementary algebra, but algebra is the stumbling-block to so many. Thus I advocate Mechanics rather than calculus as a subject for the ordinary student, and would much rather lead to calculus through Mechanics, and never get beyond the Mechanics part except with the cleverer students.

One difficulty is that Mechanics is often left by the mathematician for the science master, and *vice versa*, and so often omitted. I would firmly place the initial stages with the mathematical master, and encourage him to consort with the science master for the provision of experiments when necessary.

(2) The approach in teaching Mechanics should be by appeal to intuition first and to experiment afterwards, when necessary, rather than in the reverse order. This is equivalent to saying that the subject should be taught mathematically, and that when certain results have been established then is the time for experimental verification—if this is necessary—that is, if the pupil is not already convinced by facts in his or her experience.

Archimedes gives a logical proof of the principle of the lever. He has become aware of it in his experience, and he sets out to prove it. Far be it from me to suggest the introduction of his proof, in strict euclidean form, but it is interesting to note that historically this fundamental principle was deduced theoretically, from knowledge based on experience. What Archimedes did was to reduce the unfamiliar case—a lever with unequal arms and weights—to a familiar case: a lever with equal arms and weights. In the same way Newton at a later stage realised that the moon in her orbit dropped towards the earth according to the same law as that of the falling apple, thus reducing the explanation of the more complicated motion to that of the simple fall. Stevinus deduced the principle of the resolved part of a weight down an inclined plane first by logical deduction. Galileo considered intuitively that motion on an inclined plane was of the same kind as motion of a falling object, but slowed down, and before resorting to experiment he deduced that speed could not be proportional to the distance fallen from rest, but that a more reasonable hypothesis was that the velocity should be proportional to the time, and then from this he calculated the distance fallen in a given time, and this latter he verified by experiment. Again from his knowledge of motion on an inclined plane he deduced the first law of motion, in the special case when the gradient is zero. Experiments were needed to verify results when they had been obtained, and that is the role I give them in elementary Mechanics.

So in our teaching I suggest that results should be obtained by general reasoning and then verified by experiment. It is one of the difficulties of this course to know how much can be done in a mathematical class-room, with impromptu apparatus, and how much demands more complicated apparatus. It seems to me that the verification of the principle of moments, of resolving and compounding forces, can be dealt with in a form-room, whereas experiments on machines need the apparatus. These last should only be performed after careful preparation, the object, of finding efficiency or of verifying the linear law connecting effort and load, being laid down beforehand. Experiments with a Fletcher's trolley are valuable for ensuring that a pupil can read off velocities and accelerations from a trace. In this discussion there may be suggestions for appropriate experiments. For the next stage of Mechanics, the specialist or would-be engineer, experiments become increasingly valuable, and I believe Dr. Baggott has suggestions to offer us.

To sum up, I think that the main object of a simple Mechanics course in mathematics is to apply simple mathematics to problems within the understanding of any pupil, and in this experiments play an incidental and not a leading part.

(3) Is the concept of force more easily introduced *via* Statics or Dynamics? In the early work on Statics all the forces introduced were weights. Stevinus, for example, produces forces in various directions by means of pulley systems and weights which we would call " Heath Robinsonian ". Certainly this is the natural beginning in Statics, and it is thus possible to slip very easily from weights to forces acting in any direction, and nothing more complicated than a spring balance is needed for measuring a force. The other approach leads through kinematics to the question : what is it that causes acceleration? This can be answered by a discussion on how a train or car starts or stops—preferably starts, as the force exerted by the rail on the driving wheel is easier to explain in this case. The fact that friction is involved does not matter, as modern students are all familiar with it, and the fact that the first reference to friction treats it as a " useful " force is perhaps a good thing. The measurement of force will again come by comparison with weight. Having established that force is proportional to acceleration, it is natural to use the fact that a body falling freely is acted on by its own weight, and this produces an acceleration g, and this makes it possible to calculate the ' constant of variation ', the force being measured in the same units as the weight. The change of speed caused by a force acting for a certain time or for a certain distance has more direct interest to a student than investigation of acceleration, for he finds so many more references to change of speed ; for example, the distance in which a car can stop from a given speed. Thus in an early course more work will be done on change of kinetic energy and momentum than on the use of the equation of motion.

At some stage, definitely later, a student has to differentiate between the force such as a pull or push due to contact of one object with another, and a force such as weight for which there is no contact or visible means of application. Thus it is important for him to emphasise the fact that resultant force is proportional to acceleration, and that when there is any acceleration there must be a force to produce it. Motion in a curve provides an excellent extension and illustration of this idea. For a simple example such as an object being swung round in a circle at the end of a string the force is clear enough, and the fact that force produces acceleration helps to drive home the fact of the component of acceleration towards the centre. For a car rounding a bend the inward force is not so obvious ; rather there is an easy tendency to talk of an outward force. For this reason I do deprecate any attempt to insert an imaginary force to maintain equilibrium. It seems better teaching to encourage the student to mark the actual forces and accelerations, and apply the equation of motion, or at a later stage to equate systems of forces to the " equivalent " system of mass-accelerations. At a still later stage the student can be taught that if an acceleration is observed he must postulate a force to account for this ; or, if he gets beyond Newtonian Mechanics, account for it in some other way. It helps to imagine a frame of reference not rigidly fixed to the earth, for example, a train accelerating or rounding a bend, and ask what is the appropriate gravitational force needed to explain the motion of a freely falling object under such conditions.

This has led me some way from the original question : " Is the concept of force best introduced *via* Statics or Dynamics? " The answer is that either is possible, and for purposes of organisation in a school it is a good thing to make the two beginnings independent. But I have a definite preference for starting *via* Statics.

(4) Units. Is it better to start with gravitational or absolute units? I have paved the way for my answer to this in my last question, and I unhesitatingly vote for gravitational units. The object of a first course is to apply mathematics to familiar concepts, rather than to introduce new concepts, at any

rate at the beginning. If, then, Statics gives the beginning, since forces in general must be measured in the same units as weights, it seems to me certain that the unit must be the lb. wt. or gm. wt., emphasising from the beginning that a lb. wt. is the force equal to the weight of one lb. If this is done, rather than allowing forces to be measured in lb., then the distinction between weight and mass can be developed more easily, and from the beginning the force has its distinctive unit. When dynamics is begun and the law established that force varies as acceleration, then some will say here is the time to introduce absolute units, since the mass enters as a distinct concept to provide the constant of variation. I believe this is quite wrong and is introducing an air of mystery when it is not wanted. A student has enough to consider here, and does not want to be mystified by the idea of mass as distinct from weight before he is ready for it. In any case the equation of motion is probably passed over very quickly at this stage and momentum and kinetic energy introduced. By the time the student has worked problems with these latter concepts he may well begin to query whether weight—to be thought of increasingly as a force—is really the quantity needed in the expressions for momentum and kinetic energy. Then is the time to journey down coal mines and up to the moon, and drive home that the same object has different weight but is the same object, and has the same kinetic energy when moving at a given speed wherever it is, and hence a constant called mass as distinct from weight. And lastly, after that has sunk in, a student will think of units, and realise that lb. wt. is not in all cases the best unit for force. Even then, as far as Mechanics is concerned, he will not bother much about the variability of the lb. wt., and while being prepared to listen to talk of dynes and poundals he will see no reason for changing his own habits. That is the stage at which the elementary course will end. I regard the elementary course as partly a failure if by the end a student does not realise the difference between mass and weight, and the need for absolute units. To ask him to do this at the beginning of his course is like thrusting something down his throat which he is not old enough to digest. Mechanics is a subject with very few ideas, but these ideas take a long time to sink home with some students. Hence an opportunity to go over again some of the work done, but with a new outlook, is most valuable. Such a re-start is afforded after the introduction of absolute units for the student who is going further, and ideas of momentum and kinetic energy can be developed with the new units. Further opportunities of pressing home the same ideas, but with a new element, occur in considering the rotation of a rigid body about a fixed axis, and, when enough calculus is known, in working by calculation with variable forces. I have come to the conclusion that it is this recurrence of the same ideas under new guises which makes Dynamics an easier subject to teach at the school specialist stage than Statics. In the latter, once the fundamental methods of reducing systems of forces have been dealt with, there is little opportunity of teaching with a new cloak.

Again I have wandered, purposely, but my conviction is quite definite that the beginner can manage better with gravitational units, and those who do go further gain by the later introduction of absolute units.

PRACTICAL APPLIED MATHEMATICS.

Dr. E. A. Baggott (Polytechnic):

I have been asked to introduce, for purposes of later discussion, the subject of Practical Applied Mathematics.

Practical work in such subjects as Physics, Chemistry and Engineering is considered to be an essential part of the training of a student in these subjects.

It is my personal conviction that practical work in a Mechanics laboratory is no less necessary to the training of a student in the fundamental subject of Applied Mathematics. The student himself should receive our first consideration. I have found that students respond to practical work in Applied Mathematics very well indeed : they enjoy this work, and their interest in the subject is most certainly stimulated by it.

In building up the logical system of Applied Mathematics in the classroom, we employ ideal mathematical models. Surely it is desirable that a student should be allowed to verify for himself that there is satisfactory agreement between the results of observations of actual physical systems, and the results to which we are led by consideration of ideal systems. In the Laboratory we can focus attention on the application of a particular principle, and study it a little apart from other issues which otherwise distract attention.

Relatively simple apparatus is all that is required, and a complete course of experimental work can readily be organised. There is an abundance of suitable experiments. I like to present the student with an experiment in which it is reasonable to expect him to supply the theoretical basis for himself. It is not unusual to find that the student of Applied Mathematics at the Sixth Form or Intermediate Science stage has not carried out any experimental work on the subject, so that he is at first without confidence in his ability to supply the necessary theory, although his knowledge of theoretical mechanics may be good ; but this lack of confidence decreases as he gains experience. This linking up of the laboratory and the classroom is important, and students feel very strongly a need to see practical work in Applied Mathematics against its theoretical background.

At the Sixth Form stage we are anticipating that the student will develop rapidly a new attitude towards experimental work, and we seek to train him to work individually in the laboratory and to become self-reliant. Experimental work involves amongst other considerations :

(*a*) Tabulation of Observations,
(*b*) Systematic Calculation,
(*c*) Legitimate Approximations,
(*d*) Graphical Representation of Observations,
(*e*) Interpretation of Graphs,
(*f*) Graphical Integration and Differentiation and corresponding numerical processes ; First and Second Moments by Graphical Methods.

These selected topics are met with in other experimental sciences, and they are topics in which the Mathematics teacher is vitally interested. In fact, the Mathematics teacher should feel himself to be responsible for seeing that students should learn something of these subjects from him. Otherwise it is possible that students may receive no systematic training in subjects of great value to them in all their practical work, at a critical stage in their development. No doubt some of the topics can be dealt with in the classroom, but I believe the only satisfactory way of dealing adequately with them is in a laboratory, where the student is performing an experiment in which the importance of a knowledge of these subjects is vividly clear to him. Practical Applied Mathematics provides the Mathematics teacher with the opportunity of dealing with these subjects in their true setting.

It is clear that the teacher of Pure Mathematics is also interested in this work, for Applied Mathematics abounds in opportunities of using all the mathematics a student is capable of absorbing. For example, the use of a planimeter, or graphical integration, can provide the opportunity for the drawing of the conic sections by means of the focus-directrix property, or by means of their parametric equations ; the ellipse from $(a \cos \phi, b \sin \phi)$ and the parabola from $(at^2, 2at)$ are examples. The ideal situation arises when

the teachers of Pure and Applied Mathematics combine their efforts to making a single course of Practical Applied Mathematics of service to both and to their science colleagues.

Since this talk is to be followed by discussion, I must be brief; but I should like to consider one or two experiments of the kind which are suitable at this stage. As one is constantly on the lookout for new experiments, I hope that we may learn of some such experiments in the discussion which follows later.

[Dr. Baggott then described some suitable experiments.]

Miss Hardcastle followed the opening papers with some remarks on Mechanics in girls' schools. She said that she was afraid her experience in teaching Mechanics below the Sixth Form level was infinitesimal compared with that of Mr. Snell and Dr. Baggott, and therefore she proposed to take up as little time as possible. They all knew that Mechanics as taught in girls' schools, apart from that which was included in the Science Course, was almost exclusively confined to the Sixth Form. As far as her experience went with boys and girls, she found little difference in mathematical ability between the sexes. She had the opportunity of teaching Mechanics to Sixth Form boys and girls during the War, and, as it happened, the girls came out best, in spite of the fact that the boys had eight lessons a week and the girls four.

It had to be remembered that, until 1867, even Pure Mathematics was not recognised as a subject in the curriculum of girls' schools, and even then the shortage of qualified teachers made progress extremely slow. At the time the *Report on the Teaching of Mathematics in Secondary Schools for Girls* was issued by the Board of Education in 1911, Mechanics was not generally taught to girls. It was stated in that report that many teachers wished for an opportunity to develop with their pupils mathematical ideas drawn from Mechanics, but since Mechanics was rarely taught until the Scholarship Stage—the second year in the Sixth Form—these aspirations came to very little. On the other hand, in boys' schools Mechanics was a recognised subject in the Middle School syllabus as early as the seventeenth century. She, herself, was in the Sixth Form when the 1911 Report was issued. She wanted to take Pure and Applied Mathematics, Physics and Chemistry, but was told there were no courses in those subjects beyond the School Certificate stage. She was made to take English, French, History and Geography, and, as a special concession, was given two periods a week for Chemistry. She had no intention of taking up these subjects for her career, and so, after her homework was finished, she read Mathematics, Physics and Chemistry and attended a Technical College on Saturdays for Practical Physics and Chemistry. She gave that as an example of the difficulty experienced in some schools by girls who wished to specialise in Mathematics and Science even as late as 1911.

In the *Report on Elementary Mathematics in Girls' Schools*, first issued by the Mathematical Association in 1916 and revised in 1928, they were told that, although a mathematical course of Mechanics would not be profitable for all, girls should not leave school without some knowledge of mechanical ideas. There had been little progress in the teaching of Mechanics in the Main School since that date. Even in the Report on the Curriculum and Examinations in Secondary Schools issued by the Board of Education in 1941 it was merely stated that possibly some elementary Mechanics, including experimental work, could be attempted.

She asked to be forgiven for drawing from her own experience in the teaching of Mechanics, since she had little knowledge of what had been done by other teachers of Mathematics. In 1923 she had a Second Division Class of girls who were not particularly good at Mathematics, or for that matter at any other academic subject. They saw little point in learning the Mathematics required for the School Certificate Examination, and she therefore

introduced a course of Mechanics, which spread over three years with an average of two lessons a week. They had full use of the Physics Laboratory, and so were able to approach the work from the practical point of view. They did not present Mechanics as a subject for the School Certificate, but she thought they gained a great deal from what they had learned. They certainly enjoyed the work.

Apart from giving examples to correlate with the Mechanics which was being taught in the Science Department, she taught no more Mechanics to the Middle School until 1936, partly on account of staffing difficulties. At that time she had a very good First Division Form in the third year of its course. Without any extra lessons they covered a fairly wide syllabus. All homework was voluntary, and possibly the girls did far more than if it had been compulsory. In most years she had a section of the Upper and Lower Fifth Forms taking some additional Mathematics. She had taught Calculus rather than Mechanics, not because she thought it a good thing, but because of lack of time. Until two years ago this work was not taken as an examination subject, but simply from the point of view of general interest. She now has quite a good group of girls in the Fourth Year who are interested enough to want to do some more advanced work, and she proposes to teach them Calculus and Mechanics next term.

There was no time to go into detail, but the lines she had followed were very much those which Mr. Snell had given them. A section of Mechanics could be introduced as soon as the necessary Pure Mathematics had been revised or taught afresh, as the case may be. Composition and resolution of forces could be taught while revising the theorem of Pythagoras and its extensions; simple cases of equilibrium could be introduced with quite easy trigonometry; rates of change, velocity and acceleration could follow differentiation; graphical methods could be used as far as possible, and simple frameworks could be introduced quite early.

As far as the order of teaching the various sections of Mechanics was concerned, she entirely agreed with what Mr. Snell had said, but she had not always carried it out. She had sometimes approached it from the experimental point of view, and sometimes from the theoretical leading up to the experimental. She hoped there were some members whose experience in teaching Mechanics was greater than her own, and who would be able to give some further information on the teaching of the subject to girls in the Main School.

Mr. A. W. Siddons (Harrow) said that during his first four or five years at Harrow he taught no mechanics, but after that time he was turned on to some specialists, and he found they were always going wrong over g. Shortly after that, with some beginners, he started work entirely with gravitational units. If people would start with gravitational units it would make an enormous difference. Incidentally he thought that geometry and mechanics were the two subjects to which children came with more knowledge—subconscious knowledge—than to any other school subject. The business of the teacher was to bring that subconscious knowledge into the conscious plane, and that was the beginning of school mechanics. It was astonishing what a lot of subconscious knowledge children had. He would start—he differed from Mr. Snell in this—with the distinction between mass and weight. He would ask a boy, "If you go to buy a pound of chocolate, do you buy the stuff or the weight? If only the weight, you might just as well buy sand, which has just as much weight as a pound of chocolate." He did not make the point too seriously, of course, but he liked a boy when talking of force not to talk about "five pounds", but about "five pounds weight".

With regard to the question of absolute units or gravitational units, he had

looked up the *Shorter Oxford Dictionary*, and there they alleged that the word "poundal" was introduced only as late as 1879, and the word "dyne" in 1873, so that these were not very old in the teaching of mechanics.

Early in his career he wanted a unit for momentum, and as he talked of the "foot pound weight" in connection with work, he asked, "Why not take 'sec. pound weight' in connection with momentum?" He used this for a long time, and then in the *Mathematical Gazette* of 1914 he found a letter from the late Professor Barrell of Bristol University, which made the same suggestion. It was a fact that when he himself started teaching, it was quite unusual to speak of energy and momentum as the time and distance effect of a force. His science colleagues sometimes wanted him to start with absolute units. His answer to that was that, so far as concerned the boys he had taught, starting them on gravitational units, there was no difficulty in the transition if it was done simply. If the teacher said, "That is all wrong, do it this way," that, of course, was the wrong attitude.

In his own school days he never once drew a force diagram accurately and never used graphical methods in statics. To-day these were used a good deal, and he himself got to use them very soon. On one occasion he had a term with a lot of rather stupid beginners, two lessons a week. They took the law of moments and did triangle of velocities. The following term those boys went to an old-fashioned colleague, who started them with resolution. His colleague came to him one day, and said, "I cannot get them to resolve; for every problem I give them they give me one of your beastly graphical solutions." If he (Mr. Siddons) had been teaching them to resolve, he would have said, "I am giving you a problem which would be easier to solve by graphical methods, but I am going to show you another method which is useful, and I want you to avoid graphical methods for the moment." If that attitude were adopted they would come to the resolution methods willingly and even happily.

Mr. R. H. Cobb (Malvern) described an experiment requiring hardly any apparatus, to demonstrate the independence of vertical and horizontal motion. Two pieces of chalk were placed on the edge of a table, one being allowed to topple over on to the floor and, simultaneously, the other flicked away with velocity over the edge. It could be seen that they hit the floor at the same time.

He was not clear as to how much time should be spent in practical work in the teaching of mechanics. Every period in the laboratory was a period taken away from theoretical teaching. That was the chief objection to the practical work. He thought that Mr. Snell possibly had under-estimated the difficulties of teaching momentum and energy. Up to that stage he agreed that mechanics was the easiest of the additional mathematics subjects, but here there was a considerable difficulty.

Mr. W. F. Bushell (Birkenhead) said that in 1909 a Committee was formed jointly from the Science Masters Association and the Mathematical Association to consider the teaching of Mechanics. It was worth noting that the main complaint of this important Committee was that Mechanics was taught both by the science and the mathematical teachers entirely independently, and, in the vast majority of cases, without either of them knowing what the other was doing. There was often a lack of correlation of any sort. Arising out of this, certain schools started mathematical laboratories so that the mathematical masters might teach on lines rather different from those to which they had been accustomed. Mr. Siddons was an early pioneer, and he himself was one of the disciples of Mr. Siddons, and tried to carry on elsewhere the idea that the same master should teach both practical and theoretical Mechanics.

He had recited this little history because he greatly feared to-day that there was still, in spite of what had been learned during forty years, a lack of connection between the physics and the mathematical master, the former being concerned mainly, though perhaps not entirely, with practical aspects, and the latter wholly with the theory ; in these days of economy it was too much to expect schools to have a mechanics laboratory specially for the mathematical master, and hence close co-operation between the mathematical and science staff was very necessary.

He certainly remembered in those early days many mistakes which he himself made in teaching practical mechanics. He felt now that he spent too long on elementary experiments, and did not get on quickly enough ; it took long to carry out some of these experiments, and, after all, the time-table was limited in extent.

Well did he remember starting experimentally, first one year with statics and then another year with dynamics. Starting first with dynamics was a failure. Certainly he found statics easier to start with, as Mr. Snell had recommended in the opening paper he had given them. The reason was the difficulty beginners found in ideas of momentum and energy. The same sort of difficulty did not occur in statics.

Mr. L. W. Clarke (Regent Street Polytechnic) put forward the suggestion that in teaching Mechanics, particularly with beginners, one should start out with the intention of teaching Mechanics, and not separately Statics and Dynamics. He advocated bringing in the two branches of the subject contemporaneously, going from one to the other as the course of one's work indicated. He had found this quite the best way with both pre-Sixth Form work and with the Sixth Form class, and it solved some of the problems of teaching which had already been raised in the discussion. On the point as to which was the best way to inculcate the conception of force, he considered that there was only one. Reference to a dictionary would furnish a whole column of definitions of the word " force " which to a mathematician were of no use whatever. He had found that to define force as " that which produced acceleration " and not to allow any other preliminary definition was the best way of approach. This allowed the early and immediate introduction of absolute units, and he considered it an advantage to start with these. He disagreed with the statement that the transition from one unit to the other was easy ; he knew many students who had not the least idea of the difference between mass and weight and force.

He supported the insistence on accurate calculation and legitimate approximation which had been mentioned by an earlier speaker, and agreed that experimental work was something which allowed that to be done. He was very familiar—as, no doubt, were many of the teachers present—with the type of student who had the most expensive and precise apparatus for calculation, and was content to give one a result to within a few degrees, or to the nearest foot when two decimal places were necessary. He thought that in practice, and not only in school, there was much to be said against the provision of highly delicate apparatus for those who were not sufficiently skilled to use it. A small and cheap clinometer would often do the work for which an extremely expensive theodolite was used.

With reference to what experiments could be done in the course of one's work in Mathematics, he suggested that those who were teaching it should keep a drawer in which were placed, literally, junk—an old gramophone motor, some pieces of string, a golf ball, bits of iron tubing, as well as glass tubing, and a set of meccano. Most of the experiments required could be set up by using these, and a little ingenuity would make them suitable for most of the established principles of mechanics. Let pupils also do experiments

with common and domestic pieces of apparatus which were easily obtained. Let them find the velocity ratio and the efficiency of a car-jack or a mangle. These were examples of machines to which the mechanics they were learning applied, and which they were constantly seeing in use. That was a way of emphasising the reality and the necessity of what they were learning.

He referred to Miss Hardcastle's remarks on girls and Applied Mathematics, and said that he could not agree with what she had said. He had taught the subject to both girls and boys over a long period, and found that though one occasionally found a girl who was as good at the subject as were the best of the boys, the general standard among girls was not so good. Boys and girls might be together at the top of the list of marks ; then came a list of boys, followed by the remainder of the girls.

Finally, with regard to whether or not to allow the use of graphical methods in statics, he strongly agreed that they should be used. Those who had little or no mathematical background could understand and use a parallelogram of forces—and of velocities. Some of them had had experience during the war of teaching members of the A.T.C. and Air Force Observers, many of whom had not had the preliminary background of mathematics. They could, however, draw and use a diagram of the kind used for composition of forces and velocities, and though it might not take them very far in mathematics, it might be the only way of bringing their aircraft home again.

Mr. H. V. Lowry (Woolwich Polytechnic) said that the only point he wished to raise concerned the co-operation between mathematics and science. The physics teachers always wanted to start on trigonometry and elementary calculus at a very early stage and before the mathematics teachers had got on to those topics in pure mathematics. He had tried for the last couple of years to get applied mathematics teachers to teach elementary calculus *via* the velocity-time graph, and so on, right from the beginning of applied mathematics. Thus these teachers were not doing any of the applied mathematics which had been mentioned so far in the discussion. This method got the students used to the calculus notation early, and he was sure this would help them later on. It meant that mechanics had to be started *via* dynamics, but he thought it was necessary to go only a very short way in dynamics before going over to statics.

Mr. P. J. Wallis (King Edward VII School, Sheffield) urged that mechanics should be taught in connection with other branches of mathematics. Thus the laws of mechanics and trigonometry which were essentially concerned with the addition of vectors should be related. The addition of velocities could come quite easily with drawings of aeroplane velocities, and this could easily lead on to the idea of the components of a force. In many other ways pure and applied mathematics could be unified : thus the coordinate geometry of the circle and the mechanics of circular motion, similarly the parabola and the trajectory, could go together. At a later stage curvature could be approached through the simple ideas of kinematics.

Professor T. Arnold Brown (University College, Exeter) said that throughout his professional life he had avoided the teaching of statics at all costs, but there were certain points in connection with the teaching of mechanics which occurred to him. First of all, it had been represented to him by a pure mathematician that logically statics was simply a particular case of dynamics in which all the velocities were zero. If that was a justifiable way of representing the subject, it would suggest that dynamics be introduced before statics.

He himself felt, in dealing with scholarship candidates, that the statics questions were always found harder—not intrinsically harder, but harder in that more knowledge of pure mathematics was generally involved—than the

corresponding questions in dynamics. It might be that if they had a triangle to deal with, they had to mingle with the concepts of mechanics a good deal of the conventional theory of the triangle, and the situation became all the more complicated if the configuration happened to be more elaborate than a straightforward triangle.

Mr. Siddons had suggested that children came with more subconscious knowledge to geometry and mechanics than to any other school subject. But was there not an element of danger in that very fact? Was the subconscious knowledge they had of mechanics a safe guide? One speaker had mentioned the experiment whereby two pieces of chalk were placed on a table, one of them allowed to topple over the edge of the table and the other flicked away with a horizontal velocity. Would the pupil from his subconscious knowledge expect the two pieces of chalk to reach the ground at the same time?

His own objection to teaching statics had always been that, in looking up the books on the subject, he failed to find in the first chapter a list of axioms or assumptions which one was allowed to make, but as he went through the book he noted many conventions tacitly assumed, and he was inclined to think that one ought to be stricter from the point of view of logical presentation, and to base the work, as it should be based, on Newton's laws of motion.

He regards mechanics as an empirical subject. He believed that the laws of motion could be verified and checked by experiment, and he could not help feeling that one ought theoretically at all events to start with experiments which would illustrate these laws and convince the pupils about their truth. Would intuition recognise the existence of the *reactions* as asserted by the third law?

One final point. He hoped that those who taught mechanics in schools were able to bring the subject home to their pupils by reference to everyday life. There was no commoner object than the motor car, and a motor car was peculiar in one respect, in that the front wheels were not upright, as one might expect them to be, but inclined to the vertical. Of course, there was a reason for that; but how many pupils, if asked the reason, could give an intelligent reply, even after two or three years' study of mechanics? During the war this question was put at the *viva voce* examination to certain candidates for commissions in the Royal Engineers. It was, he supposed, a fair question and he invited their attention to it.*

He felt that this was a most interesting discussion. The question of the best way of teaching applied mathematics was a vexed one. London University had apparently taken the line of least resistance and for the Honours degree had dropped the subject of statics altogether.

The President said that it was made abundantly clear in the *Report on Mechanics* published by the Association some years ago concerning pupils coming along with a good deal of experience, that it was the business of the teacher to take advantage of that experience and to clarify it.

As regards statics and dynamics he would not wish to take sides, but if statics, as one speaker had suggested, was to be considered as a particular stage in dynamics it ought to relate to the case when the acceleration was zero, not when the velocity was zero.

* If r be the radius of either wheel, θ its inclination to the vertical when the car is travelling in a straight line and ϕ the angular deviation of the path of the wheel from this line, the vertical height of the centre of the wheel above the ground is given by the expression $r \cos \theta/\sqrt{(1-\sin^2\theta \sin^2\phi)}$, which represents a steadily increasing even function of ϕ. It follows that, when the car is steered round a bend, the height of the front axle is slightly increased, and hence, owing to the effect of gravity, the car will tend to "straighten out" automatically after the turn.

Miss Cook (Edge Hill) said that when a student came up, having already done mechanics, she was usually very disappointed when the College did not go on with it. Unfortunately, they could very rarely go on with it, because they were in a small class of six or eight people, half of whom had done mechanics and half had not. As a rule, to make a course which was profitable to all, they had done new work in some other direction and encouraged the girl who had already done some mechanics to continue by private reading.

As to the controversy between statics and dynamics, it was worth remembering that the attention was invariably drawn to something which was moving. It was much more natural to look for the cause of movement than for the reason why something did not move. She thought teachers did sometimes find it difficult to make the student understand how things were in equilibrium, what was keeping them in equilibrium. It was easier to show what made the machine work, and the reason for the velocity ratio, which was the most obvious thing about it, rather than to arouse interest in the equilibrating forces which kept it steady. That, at least, was her impression of it, and she thought most pupils would be more prepared to look for the cause of movement than for the cause of something standing still.

Mr. Sampson (Ealing) said that he agreed with Mr. Siddons and Mr. Snell that gravitational units be used first. Difficulties there only arose when students encountered the force unit in the metric system, the scientific unit, for which the scientists had automatically used the dyne. He suggested that absolute units be introduced to the students at that point.

One other point which Mr. Siddons had mentioned was the unit for momentum, and another speaker had referred to the difficulty which some students had in the conception of momentum and energy. He himself had found that the energy conception had been fairly readily understood, because energy was capable of being stated in fairly concise terms, whereas momentum did not seem capable of being defined in terms so precise. On searching the textbooks for definitions one came up against *mv* as the definition of momentum, and he found that many students wanted something else. If one went back to Newton one found only that momentum was " quantity of motion ".

Mr. A. W. Siddons (Harrow) said that Mr. Cobb had stated that he had difficulty in getting on to momentum and energy. Momentum might be defined as the amount of impulse to be exerted on a body to bring it to rest. Whole lists of examples could be given. A force acted on a body for a certain distance or a certain time. They had to find out what velocity it would acquire or what velocity would be destroyed, and he would go through the various examples, asking whether each was a question of energy or of momentum. If time came into the question, then it was a case of momentum ; if distance, it was a case of energy. In that way they could be got to appreciate the point very quickly.

When he started teaching he had boys who had a considerable experience of dynamics, but he found that they never used momentum or energy.

Mr. Todhunter (R.N. College, Dartmouth) said that at the R.N.C. the subject of Mechanics had been taught in turn by the scientists and mathematicians. Recently, however, the two departments had got together and hammered out an agreed syllabus and treatment. He himself, having been brought up on mathematical lines, preferred the absolute unit and the theoretical approach, but he had come to the view that, except perhaps for specialists, the practical approach was the better one.

At his college they had three periods a week, which included one double period a fortnight in the experimental laboratory. The boys went into the laboratory, collected their apparatus, and got on with the job. They had a book of instructions and had to think out for themselves what was involved,

with, of course, the opportunity of reference to the master. The experiments conducted were usually quite simple.

Mr. J. T. Combridge (King's College, London) said that some of the difficulties mentioned in the discussion often arose from the Physics teacher's having to cover, under the heading of Mechanics, the Applied Mathematics syllabus in a third or a quarter of the time required by the Mathematics teacher. No wonder that the Physics teacher could not keep step, but had to go on faster than the mathematical colleague.

The speaker agreed with Mr. Snell, but some later speakers had strengthened his fear that perhaps Mechanics in schools was being killed by Applied Mathematics. It was so difficult to get pupils to work simple examples from first principles. Their one idea seemed to be to find the right formula and to put in the appropriate numbers, while statical forces were for them geometrical rather than physical concepts.

He had often had to take a class in Applied Mathematics for Intermediate Science, of which perhaps one quarter had already passed in this subject at a Higher Certificate examination while another quarter was practically beginning it. In such cases he had gone back to some of the old pioneers, such as Galileo and Stevin, indicating hypothetical experiments on the blackboard and trying to concentrate on principles. In this manner one could go quite a long way, and even introduce simple problems on work and energy before coming to mass or to regarding acceleration as proportional to force, and, by talking about " load ", avoid many difficulties connected with the idea of weight or with units.

What Mr. Siddons had said reminded him that there were two warnings to be uttered in connection with all this. Once he was getting on quite nicely with this work and energy business when one lad, who had " done " applied mathematics at school, said to him rather suspiciously, " I was brought up on Newton's laws of motion. Why mayn't I do it that way? " He had replied that that was an important way of doing it, but that he wanted him to try it in this other way instead. Later he started getting together some notes on this subject for the *Gazette* and possibly for a book, when he remembered having seen a book which had some useful indications along these lines; he discovered not only that the textbook had been written already, but that it was the one on which he himself had been taught mechanics by his physics master at school. This seemed to show that Mr. Siddons might be very right in saying there is a good deal in the subconscious.

The teachers of geometry had recognised that the child had got a certain intuitive geometrical knowledge, and had shown how he might be taught geometry by canalising and directing that subconscious knowledge in the right way. He believed that something similar needed doing for Mechanics and Applied Mathematics. There was still a great deal to be done, and it was for the applied mathematicians in the Association to go on to correct that subconscious intuitive knowledge of Mechanics where it needed correction, and to use it as a basis for teaching the quite distinct subject of Applied Mathematics. But it should be recognised that Applied Mathematics was a formal discipline just as much as geometry, and that just as less able pupils who will never do much at Geometry may be quite good at Mensuration, so pupils who may never be able to cope with Applied Mathematics may be given quite a sound training in Mechanics.

Mr. K. S. Snell, in reply, said that he thought Mr. Todhunter's real point, in connection with the doing of the experiments, was in the method of approach. Should the experiments be done first and the pupils then be asked why a certain thing happened, or should the pupils be asked what they would expect, and then carry out the experiment to ascertain if they were correct? He

thought the latter, so that a boy knew what he was trying to verify by the experiment.

With regard to what Mr. Siddons had said, he himself would work from things which were moving, from a given velocity to rest, and then he saw no more real difficulty in dealing with momentum than with kinetic energy. Both represented something about an object, and one needed to find either what force would stop the motion within a given time or within a given distance, and, as Mr. Siddons had said, one used momentum in the one case and kinetic energy in the other.

Mr. Clarke had said that he preferred to take dynamics first rather than statics. The difficulty about that in practice was that the more difficult ideas arose more quickly in dynamics. It was much easier to deal with the resolution of forces than it was with the problem of momentum and energy. That was one of the reasons why he preferred statics first.

When they go on to the later course a great deal of what had been said applied to the Sixth Forms. This was the time to start with vectors and to deal with the laws of motion and with statics as a particular case of dynamics, at any rate as far as the particle was concerned. This would prove too difficult for a rigid body.

He was grateful for the comments on his opening and he remained unrepentant about units in the preliminary course.

CORRESPONDENCE.

THE TEACHING OF MECHANICS.

To the Editor of the *Mathematical Gazette*.

SIR,—In the discussion on the above subject at the Annual Meeting, January 1950, there appeared to be two schools of thought on the question of units, one in favour of using gravitational units at first without too much insistence on accurate terminology, and the other in favour of starting with theoretical units. As far as I remember no reasons were given except " usage ". It seems to me worth considering whether to allow (I do not say to encourage) pupils to use the same word to express a mass and a force, for even part of their course, does not make it more difficult for them to realise the difference later.

The following example shows the sort of confusion which does actually occur frequently.

" A block of mass M gm. rests on a rough horizontal table and the coefficient of sliding friction between the block and the table is μ. Find what horizontal force must be applied to the block so that it shall move with an acceleration of a cm. per sec. per sec."

First solution. Friction $= \mu M$ gm.

Required force $= (M \pm \mu M)a$ or μMa,

and the unit here may be gm., gm. wt. or dyne.

Second Solution. Friction $= \mu M$ gm.

Force required to produce acceleration $= Ma$ gm. wt.

Hence total force $= \mu M + Ma$ dynes.

Yours, etc., J. J. WELCH.

MATHEMATICS IN THE COMPREHENSIVE SCHOOL.*

Mr. F. J. Swan (West Norwood Secondary School) : It is always with feelings of awe and wonder that I face the distinguished audience of brilliant mathematicians and/or brilliant teachers of mathematics which gathers for the Annual General Meeting of this Association. As I can lay claim to neither type of brilliance, the programme committee in its wisdom is careful to choose for me a subject which will conceal my ignorance of mathematics beyond school certificate stage.

In 1946 I was privileged to open a discussion on " The Place of Mathematics in Secondary (Modern) Schools ". To-day, it again falls to my lot to plead the cause of the *ordinary teacher* of the *ordinary child*. (I doubt, however, if our colleagues from grammar schools would agree to call some of our children " ordinary ".) May I say how glad I am to have with me my colleague Miss Y. Giuseppi, who is actually teaching mathematics in a newly-formed comprehensive school, to tell of her classroom experiences. Our subject is, according to the programme, " Mathematics in *the* Comprehensive School " : it will probably resolve itself into " Mathematics in *a* Comprehensive School "—we are duly humble at West Norwood. When I last spoke in this hall to a similar meeting I had been on the staff of an emergency training college for about eight months, and before that I had been mathematics master for twenty years in a large grammar school for boys. I had not taught in a secondary modern school, for taking over a class for a student cannot be called teaching. However, the visits to schools of all types gave me an opportunity to ask questions and observe teachers and pupils and school and class conditions. I had, in common with many of my colleagues, for some time been uneasy about the mathematics taught in grammar schools—not more than two-thirds of our pupils could cope with the formal work needed for the examinations—the remaining third floundered unhappily, but even so some of this residue scraped a pass in school cert. My experiences as an examiner in the Junior County, Supplementary County, School and Higher School Certificate examinations did not lessen my uneasiness, and I viewed with alarm the attempt to foist on to the other types of secondary school a watered-down grammar school scheme.

During most of 1949 I was making plans for the comprehensive school at West Norwood, and I feel that I had not overestimated the difficulties—quite the contrary, in fact, and although my views did not coincide with those of my former colleagues, I still hold obstinately or tenaciously, as you will, to most of what I said in 1946.

The secondary grammar schools owe much to the public schools, but the traditions of the latter were passed on by masters who came from them to take up responsible posts in the grammar schools. It must be pointed out, however, that the pupils were of the same kind : the difference being a matter of degree only. Formerly, the aristocracy or the wealthy sent their sons and daughters to public schools—the less wealthy to the grammar schools. To-day, the independent schools claim the A's—the grammar schools the rest of those who " pass the scholarship ". The staffs, too, were similar in that they were mainly graduate—those who taught in public schools were mainly from Oxford or Cambridge and very often had not been trained as teachers—in fact, a teaching diploma was often suspected to be cover for a poor degree.

In the comprehensive schools the child population consists of those who do not secure admission to the grammar or independent schools. Theoretically, the comprehensive school caters for all ranges of intelligence, but

* A discussion at the Annual Meeting of the Mathematical Association, 4th January, 1950.

there are still some unenlightened parents who would rather send their boys and girls to grammar schools. The staffs, too, are in the main non-graduate —men and women trained in the two-year colleges or, in the case of many new entrants, in the emergency colleges.

Quite recently, it is true, there has been a slow but steady flow of graduates, some with good grammar school experience, into the comprehensive school. There are, however, in the main few points of resemblance between the grammar schools and the comprehensive schools. While it was, therefore, quite reasonable to argue from the public school to the grammar school, I hold that it is fundamentally wrong to argue from the grammar school to the comprehensive school. I make no apology for this rather long preamble, for I am anxious that the stream of discussion shall not flow into unprofitable channels as it did in 1946—at least some of us felt that way about some of the points raised. Mr. Riley of Wolverhampton was good enough on that occasion to epitomise my remarks by fitting a text—" the child in our midst ". I should like to take a similar but more definite text to-day—" the comprehensive school child in our midst ".

Perhaps a short account of the school at West Norwood will help to focus attention on the essential difficulties which confront those who are conducting the " comprehensive experiment ". There are approximately 1,100 pupils, of whom rather more than 600 are girls, and the girls are, on the whole, of rather higher intelligence than the boys. This is a local factor, depending on the number of grammar school places available in the neighbourhood. The intelligence distribution shows three variations from the normal. Firstly, as I have said, we have no A's in the first year. The A's in the higher classes are those who for reasons best left unstated have been promoted to West Norwood from the grammar schools to which they first went. Secondly, the distribution is bimodal—there is a mode at the upper end of the central school range and also at the upper end of the modern school range. The reason in each case is clearly that parents hope that if their children show promise they will gain promotion to the next grade. This promotion incentive if kept in its proper place is valuable, for we at West Norwood are neither foolish nor high-minded—we do not ignore the profit motive. The third variation is at the lower end. Our school is in what is known as Division 8, and bordering as it does on two other divisions it is a convenient dumping-ground for their unwanted E's.

This year, the first year of the amalgamation of a two-stream mixed central school with a three-stream modern school for boys and a similar modern school for girls, the grouping is as follows :

1st year—eight streams	11, 12	- - - -	18
2nd year—eight streams	21, 22	- - - -	28
3rd year—seven streams	31, 32	- - - -	37
4th year—eight streams	41, 42	- - - -	48
5th year—two streams	51, 52		

We have a staff of 54, some part-time, of whom only three devote the major part of their time to mathematics, and of these two only have followed a graduate course in the subject. Thus at present rather less than half of the teaching of mathematics is in the hands of specialists ; of the remainder, some teachers take two groups for mathematics, while others take the subject with one group only, *i.e.* the group which they also take for English and some other subjects. The use of staff is a matter which calls for carefully planned but bold experiment. One of the fundamental aims of the comprehensive scheme is to give the widest possible choice of subjects, and further, to allow the study of the subjects to be pursued at the right level in the appropriate

manner and at a suitable rate. At first sight this seems to suggest that there will have to be increased specialisation. We should not assume, however, that the specialisation must be as narrow or as intense as that in the grammar schools. Group specialisation may replace subject specialisation. Mathematics, linking as it does with so many subjects, will, I think, lose much if it is always linked with the same subject, *e.g.* mathematics with science. It may be found profitable to ask one teacher to specialise in mathematics and geography, a second in mathematics and light craft, and so on. Miss Giuseppi will tell you of another method by which she has tried to break down subject barriers.

It will doubtless be pointed out that this new outlook on mathematics teaching together with the extension of the syllabus consequent on the raising of the school-leaving age will make too heavy demands on teachers trained in other days and in other ways. It is possible to release teachers for half a day a week to attend special courses on P.E., folk-dancing, visual aids and the like. Could not the authorities arrange similar courses in mathematics? This suggestion seems much too reasonable to be acted upon. In a large school it is possible, too, to arrange sectional staff conferences, and we hope to adopt this method at Norwood in the near future. This latter plan can only work in a large school where it is possible to have something of a specialist in each branch of study. I cannot emphasise too strongly the point that all planning in mathematics must take account of the staff, and further that those who wish must be given the opportunity to become acquainted with new developments. We are anxious to stimulate interest and to encourage initiative in our pupils—we must first ensure that our teachers are interested, and that they are encouraged to experiment boldly and wisely. I therefore throw the first bone of contention to the meeting—the staff.

I want to pass next to the pupils. As I said earlier, the comprehensive school should cater for all ranges of intelligence. At the moment the higher ranges are missing, and I will therefore consider only the distribution as it is and as it is likely to remain for years to come. Quite apart from the fact that comprehensive schools have to establish and justify themselves, it will be many years before they can be properly housed and equipped. Parents are hardly likely to send their boys and girls to a school such as ours which does not possess a square inch of playing fields. The girls cannot play hockey, and the boys play football on a dry playground without proper markings and with their coats for goal-posts—our school, too, is in a neighbourhood where there is a superabundance of playing fields used by grammar schools and private clubs. To return to the children—our top stream, then, would correspond roughly to the grammar school C stream. The other two selective streams will contain many pupils only slightly less able because of the bulge at the top end of the central school range which we have already noted. The modern school intake contains one very good group judged by modern school standards together with four other groups, the lowest of which contains rather more E's than usual, for a reason stated earlier.

Before turning to the general plan of the mathematics course we may look at the other end of the school career. What do the boys and girls want to do when they leave school? What are the parents' wishes? Considering first the upper streams—few parents have any desire to send their children on to the University, and those who expressed to me their regret that their sons and daughters failed—always this word failure—to obtain admission to a grammar school are much more concerned with the social standing of the school than with the type of education given. The social and economic and emotional factors are most important in our work. The fact that we have not to cater for University entrants leaves us very much more freedom in

our selective streams than is possible in the grammar schools. For instance, girls entering nursing or boys taking clerical posts may need a certificate of general education, but only in cases in which mathematics is an essential part of the equipment for a career need the subject be taken in the certificate examination. Thus only those pupils who can really profit from a school certificate course in mathematics will be expected to follow it. The decision will, of course, depend on a number of factors—the ability and aptitude of the pupil, the opportunity for developing and using this ability and aptitude and the desired career—to name a few of the more important. When candidates are entered for mathematics in the certificate examination it will be on the alternative syllabus as at present.

At the other end of the scale the boys will probably become manual workers, railway porters, or the like, while the girls will take up domestic work or work in local shops. I will not enlarge on the mathematics planned for these pupils, as Miss Giuseppi will tell you what she has managed to achieve in the short space of one term. Perhaps I may just call your attention to some conclusions drawn by Professor Schonnell about the effect of choice of work on success obtained. It was found that when the arithmetical work was based on the actual experience of the children, that is, when the problems were within the knowledge of the children, there was an all-round improvement of 45 per cent, but that in the case of the lower intelligence ranges the improvement rose to the remarkable figure of 75 per cent. I will leave my colleague to expand this point.

And now to pass to the actual planning of the work in mathematics. There is one type of syllabus which is fatally easy to devise. May I give you a sample :

1st year A—A's Arithmetic to page x,
B's Algebra to page y,
C's Geometry to page z.

1st year B—as for A, omitting harder examples in Arithmetic and Algebra and formal proofs of theorems p, q and r.

1st year C—As for A, using only the simpler examples in Arithmetic and Algebra. Formal proofs to be insisted on in cases of theorems l, m and n only.

2nd year A—Revision of 1st year together with ——, and so on.

Of course, nobody here could possibly be guilty of devising such a syllabus, but I assure you that such syllabuses do exist—and are followed. I heard of one extreme case in which the master dictated on the first day of each school year the homework for the year and the dates on which each exercise was to be handed in.

And now I toss you my second large bone—"the common core". In 1946 I expressed doubts about the common core ; since then my doubts have greatly increased, and I must now say that I feel the existence of such a core unlikely ; in fact, had I a little more courage I would say there cannot possibly be a common core in the mathematics scheme. I will now try to give my reasons.

Firstly, may we consider for a moment three aspects of a number fact :

(1) The *existence* of the fact:

$$5+3=8\ ;\quad a^2-b^2=(a+b)(a-b),$$

which are facts, whether the teacher teaches them or not or whether the pupils learn them or not.

(2) The *statement* of the fact—the method of statement is learnt by the pupil from the teacher and is merely a matter of convention. A parrot may learn a statement.

(3) The *consciousness* of the fact. It is at this point that intelligence begins to play a part. Consciousness does not follow automatically from the first two aspects.

Have we, as teachers, paid too little attention to this third aspect, I wonder? It is because of this aspect that I feel we must move right away from the idea of a common core in mathematics. The lower the intelligence the more limited the range of consciousness and, indeed, the fewer the number of facts that can be learnt. Even some parrots have more limited vocabularies than others—fortunately, perhaps. Time is short and I cannot do more than suggest this line of thought. With it I should like to couple another. There will, I hope, be universal agreement with the statement that the work presented to the child, group or class can only be considered appropriate if it affords an opportunity for success at every stage. I hope the key-note of our work will be happiness and success, for I am convinced that in mathematics nothing succeeds like success.

Considering these points I am forced to the conclusion that to each child we should present at any time only the facts which he is capable of learning and converting into consciousness. Thinking once again of the range of intelligence in our first year intake, it is clear that the mental age-gap between the brightest and the dullest is at least three years. To insist on a common core implies clearly that all but the dullest must learn much less than they are able and at a much slower rate. I wonder whether grammar schools with three or four streams ought not to pay much greater attention to these considerations.

I want to make it clear that I am denying only the possibility of a common core in mathematics. I understand that attempts have been made to devise a syllabus in other subjects with some features common to all streams. Thus, in geography, for instance, all groups would study coalfields at the same time—the dullest putting blobs of paint more or less accurately on maps to indicate the distribution of coal in England say, while the brightest would consider distribution and its economic implications. I am afraid that this is a common core more in appearance than reality—but I know less even of geography than mathematics, so I will not proceed further.

The desire for a common core seems to arise from two considerations. Firstly, special aptitudes do not become apparent or subject to test in the average child before the age of 13 and, therefore, it is argued specialisation should not be attempted before 13. By a shocking piece of logic this is distorted to mean that all children must do the same things until they reach the age of 13. This seems to me to be utter nonsense. Secondly, and perhaps more strongly put from the grammar school side, is the argument that if all children follow the same course from 11 to 13 it will be easier to transfer any bright boys or girls who were missed by the scholarship examination at 11. Thus, 80 per cent of the total school population must be condemned to a completely unsuitable education for the sake of 1 or 2 per cent. Is this reasonable or just? In any case, if we pursue the argument that the number of grammar school places must be increased until it is certain that no suitable child is excluded, are we not in principle pledged to the comprehensive school plan?

To return to the question of specialisation. It is essential, I feel, to redefine this term. In our scheme of things we hope to meet the needs of all our pupils throughout the whole period of their school course. Thus, from the point of view of the child he is a specialist from the start in that the course he is pursuing is special to him. The question of dropping this and taking up that will not arise, for the approach will throughout be positive. Specialisation, then, can only mean for us the provision of a sufficient variety to satisfy

the needs of all our pupils. A tall order, you say? Indeed it is, but it is the goal towards which we must move. What of the administrative difficulties? The reply to this question is simple—every administrative problem has its solution, and no such difficulty should be allowed to become a barrier to educational progress.

Continuing with the idea of specialisation, it is clear that when aptitudes become defined the courses offered in mathematics must cater for these new appetites. It is our intention to submit all pupils to a special series of tests towards the end of the second term of their second year, *i.e.* when they are roughly 13 years of age. The results of these tests together with any expressed wishes of the parents and/or pupils will decide the type of course recommended. It is, I think, in accordance with the modern tendency to avoid a sudden break at 13. Mathematics should not suddenly change to commercial arithmetic and book-keeping, nor to technical drawing and technical calculations. There should be a turning towards commercial or technical ideas, but bearing in mind that our aim is a sound general education we shall devise for our third-year groups a course of mathematics flavoured with commerce rather than one of commerce (*alias* shorthand and typewriting) flavoured with arithmetic—a course, at present, all too common.

In the fourth and fifth years we visualise a rather more definite bias for those who will be likely to profit from such special courses of study and whose aptitudes have been established. There will, of course, be " unflavoured mathematics " available at all stages.

The most troublesome task before us is to find the suitable topics for each mental and physical age. Much research has been done already, and I hope my mathematical colleagues will gather together the results and if necessary undertake further research. A concise record of mathematical matter and method is sorely needed. When compiled it will give to each teacher an indication of the mathematical food suitable for each group, and it will be the task of each individual teacher to draw up attractive and satisfying menus. In this way we shall, I feel, whet the appetities of our pupils and satisfy them. Too long the question on the lips of too many teachers has been " What textbook shall I *follow?* "

Miss Giuseppi will give her views on the textbooks we have inherited. What a lovely bonfire many of our mathematical books would make—they are so dry! I have some ideas of what I would like our new textbooks to be, but for obvious reasons I feel it is not wise to broadcast them yet.

There were many more things I wanted to say, but I must not encroach on my colleagues' time. May I in conclusion say one more thing that has been much on my mind. It concerns repetitive work and activity. I have been watching with some interest the work of one of my colleagues who has a dull group. Whenever I pass or enter his classroom the boys and girls are obviously happy and busy. The class is usually working in groups, and each group has its collection of sums to do. The groups come up to the teacher to have their work marked, and there is obvious excitement as the neat R's pile up. I puzzled over this phenomenon for some time. My doubts were resolved by a personnel manager in a local factory in which I spent a whole afternoon. I asked why the men and women did not seem bored with their work, and why they piled their work up instead of sending it along on the conveyor belt. I was told the staff tried various jobs until they found one they liked, and that the work was allowed to pile up in front of them, as it was found that output was much improved when the results of their efforts could be seen. My colleague was right—he had chosen a method he liked : he was choosing suitable work for his pupils, and both he and they were enjoying success.

They were active. There are many forms of activity, and I am old-fashioned enough to believe that disorder and activity are not synonymous. The key-note of our work, then, must be to treat every teacher and child as an individual, to give freedom to all, to encourage initiative, and to let all be truly happy in the enjoyment of their success.

Miss Y. Giuseppi (West Norwood Secondary School): I feel it is a great honour to be asked to speak to you to-day, but I must tell you that I am really the unfortunate victim of that old Army technique which begins: "I want three volunteers . . . !" Need I say more?

The problem of the mathematics teacher in a Comprehensive School is a transitory one. By that I do not mean that it can be solved in a short time, but that each year, for four or five years, a fresh set of complications present themselves. I do not know to whom we are indebted for our particular Comprehensive School, but how much easier would have been the task confronting us if we had been able to commence as we hoped to continue. For we have not only to decide what is to be our aim and plan of campaign for the 300 children in the eight first forms, but we have also to consider how much mathematics can be attempted with the 800 or so other children who have already done some form of *Arithmetic* for the last one, two or three years. I stress the word Arithmetic, because I am sure that in most schools of the Secondary Modern type the Mathematics syllabus consists of very little else.

Let us even suppose that we have decided upon eight suitable syllabuses for the eight streams of our school—for I do not subscribe to the idea that for two years it is possible to have a "common core"—even then, our task is not over. For the type of work that will suit this year's 1.3 may not necessarily suit next year's, or last year's. Hence I am sure that we must study the particular group of children concerned, consider their needs and ability, and adapt our methods and ideas to suit them. In this type of school more than in many others we are free to use our own initiative and imagination in the matter of the syllabus, since few streams, beyond the 1st, 2nd and possibly 3rd, are dominated by a Public Examination. Provided, therefore, that our ideas do not clash too violently with the powers that be, I think that the job of the mathematics specialist can prove most interesting.

I am particularly concerned with the teaching of mathematics to the *dull* child, the child whom most of my distinguished audience never sees. I have often heard Secondary Grammar School teachers despair of their C forms, and I have even despaired of a C form in a Grammar School myself, but I am sure that you cannot even imagine how scintillating are your pupils, until you have made the acquaintance of our eighth stream! Since well over 80 per cent of a Comprehensive population is non-grammar, it is well to concentrate on them, for the path of the others is well trodden and clearly defined.

My aim in dealing with the duller child is to capture his interest, so that he will enjoy such mathematics as he is capable of absorbing. And to this end I devote all the coloured pencils and paper, paste and cardboard, that I can legally, or otherwise, acquire. For what many children are incapable of expressing in words or figures, they can often show with their hands. They cannot define a right angle, but they can point to objects which contain right angles. I feel, too, that colour attracts most children, so I do not hesitate to brighten exercise books and blackboards with coloured shapes and diagrams. We all know the thrill of receiving a parcel wrapped in gaily-coloured paper, tied with festive ribbon, and peppered with jovial seals, how much more exciting it looks, even to adults, than the conventional brown paper from the bookshop.

Textbooks in other subjects are being made more attractive for children. We even find pictures and puzzles in the Latin Grammar and French Primer,

while the Science books bristle with things to " make and do ". But what of our Mathematics books—they present on the whole a sorry sight. Our school, before stock-taking, was blessed with what I should imagine were the first editions of almost every Arithmetic book which has been published before about 1920, and I am sure that there was not *one* picture or *one* trace of colour in the whole collection. I must in fairness add that I have seen some attractive modern textbooks, but so far none of them caters for the lowest streams in our type of school. For the duller child needs problems with which he can cope. He needs to see the reasons for learning a particular process before he will apply himself. He needs to learn the mathematics to fit him for his everyday life.

There is a current idea that unless a child is capable of performing arithmetical techniques correctly, it is useless to introduce him to mathematics. I think it depends on what you mean by mathematics, and on how *much* mathematics you want him to learn. All children acquire a space consciousness before a number consciousness, and *both* must be developed by our teaching. Even the less able child can appreciate a considerable amount of informal Geometry, and if we spend his time on this subject, rather than in " hammering home the Arithmetic ", I do not think it is to his disadvantage.

I do not, however, mean that any deductive work can be attempted, or any formal propositions given to any child below our third stream, and I should delay this until the third year. But the others can learn the use of tools : the ruler, a pair of compasses, a simple protractor, and with the more intelligent third and fourth years, the use of clinometer, theodolite and surveyor's chain. If, for our non-grammar children, there was in existence a graded set of rulers, it would be a great help. We could start with one marked in inches, then one marked in inches and halves, one in inches and tenths, etc. At present the six or seven fractions on a single ruler tend to confuse the backward child. The usual protractor, too, is too involved for some of our children. We have made our own in several classes, using set squares. We begin with one marked from left to right from 0° to 180° going up in 15°, later we make one from right to left from 0° to 180°, and then we are ready to use the final combination. This may seem a good deal of trouble, but to children in fifth, sixth, seventh and eighth streams it is very necessary.

My most difficult class is 4.8 because they know very little Arithmetic, are extremely apathetic, and are most of them anxious to leave school. Fortunately, as our fourth year does not consist of mixed classes, it is fairly easy to cater for them. I do not think that I have taught them any more about the processes in Arithmetic, at which they have been working for the last three years, but I have tried to broaden their vision, and make them like Mathematics a little better. They neither knew nor liked the multiplication tables, but they did like finding out what coloured pattern each table produced. They also liked making and using Napier's Bones, and spent a long time working multiplication sums to check their calculations, just in case Napier did not cover every possibility.

With this particular group of girls, I have had the opportunity of linking up with the Domestic Science teacher. She was in despair because these children, when told at 11.25 a.m. to put a batch of cakes in the oven for 25 minutes, had no idea when to take them out ! Neither could they adapt a recipe making twice or half the quantity, or find the cost of a meal. This gave me plenty of material to start with, and the girls were certainly interested in eating ! My first-year class had made a set of clocks for Geometry, so we used them to work out the time for various recipes. It may surprise many people to learn that even with a clock on which to count the minutes, many

of these fourteen-year-old girls made mistakes ! Then we collected gaily-coloured recipes from magazines, pasted them on cards and proceeded to get through a considerable amount of work, halving, doubling and trebling lbs. and ozs., qts. and pts. When we came to pricing the items, however, they did not find it so easy. They collected the prices of the usual groceries from the shops and made a class list : from this each girl priced her own recipes. This type of work means more trouble in marking than turning up the answers to Ex. 21, but the girls enjoyed it, and could really see the point of Arithmetic applied in this way.

We have also combined with the Needlework Mistress, who wanted the prices of various garments worked out. A hundred or more specimens of materials were pasted on cards with the price per yard written underneath. The girls worked in groups, choosing a suitable material for each garment, and working out the cost.

The Art Mistress then gave us an opportunity for revising what little we knew on fractions and scale-drawing. She is taking a course on colour and design in houses, and wants to make models of rooms and furniture. We first learnt to use rulers correctly, and then drew many plans of everyday objects. The model-making will come next term.

With thought, too, of our prospective waitresses, or perhaps it was for the customers, we made a set of menus and worked out the cost of many different types of meals. As far as Geometry goes, we have attempted very little. We have considered symmetrical shapes, talked about rectangles and squares and right angles in connection with models, learnt to use compasses, and have drawn patterns based on the circle, but little else has been possible in view of their short time left in the school. Next term I hope to teach them how to read a railway time-table, a gas and electricity meter, and a ready reckoner, also the intricacies of the hire purchase system and how to interpret pictorial graphs. If they can do these things by the time they leave I shall be happy, and I shall not worry about the fact that they would not even know where to start a sum about two taps filling a bath with three holes in it and the plug out !

In dealing with the fourth year, my chief problem has been what is the best section of Mathematics to present to them. My other two classes in this year, one a boys' and one a girls', consist of the brightest children in the Secondary Modern streams, and they are capable of quite a large amount of practical Mathematics.

With the boys, I have devoted some time to out-of-door work. We have made simple clinometers, and have used them to find angles of elevation—I wouldn't let them risk their necks, when they offered to follow up with angles of depression as well ! Later they found the tangents of various angles by means of drawings on squared paper, and eventually the class worked out its own table of tangents correct to two decimal places, of every degree from 0° to 90° going up in 5°'s. Using this table they worked out the height of the school, and several other objects. Once the boys realised practically how tangents were useful, they were quite keen to go on, and used the 4-figure tables to work out some more involved problems. With this class I have also tackled logarithms and some simple surveying, and they are very anxious to master permutations before they leave !

The fourth year girls' class has had rather an over-dose of graphs. I should not *plan* to take graphs *en bloc* in this fashion, but as they had not done any before, and were interested, we continued. With this particular class I have endeavoured to make them " group conscious ", because I think that a great deal of this practical Mathematics must be done in sets of five or six. We started quite simply by finding out the height of each girl in the group and

drawing a graph. Later we showed pictorially how many children were in each family, how many pints of milk they consumed daily, etc. We should have weighed ourselves on the extremely complicated school scales, but it took me nearly ten minutes, and I was so dissatisfied with the result, that we omitted that graph. Borrowing an idea I had heard at a lecture, we went out in our groups, and counted all the buses, cars, bicycles, pedestrians, etc., passing six vantage points near the school. On our return we made a composite graph of our results.

The third year is not so difficult to plan for, because it is possible to follow a two-year course, and more Mathematics can be attempted. I started with exercises in measuring in several types of units, sending groups into the playground for practice in yards. Then we experimented with circles and circular objects of all kinds, measuring their circumferences and diameters with cotton. From these we calculated π, and found the formula for the circumference of the circle. Most of our problems we then worked practically and checked by calculation. One of them was counting how many times a bicycle wheel turned whilst being taken across the playground. The next topic was area, which I combined with scale-drawing, and the class found the area of Australia and Great Britain : fortunately, our results kept reasonably close to the information given by the Geography Master. All the problems on the areas of squares, rectangles and triangles concerned the class, the home, or the school.

My second and first year classes both consist of bright children, some of whom, in the first year at least, are capable of studying Mathematics for an examination. As the second year group have only done Arithmetic before, they are covering the same ground in Geometry as the first year, although, of course, they are much quicker. Most of the children in the first year group were concerned about the mysterious word on the time-table, "Mathematics", and it was in an endeavour to make them enjoy their first taste of the subject, and feel able to do well, that made me neglect their Arithmetic for the first week or so, and concentrate on Geometry, which they all felt able to master. They drew, coloured and cut out symmetrical shapes and patterns, extending this later to three-dimensional considerations. Then they made protractors and clock faces, and worked out many angles, leading up to the fact that three $\angle$'s of a $\triangle = 180^\circ$. This we proved to our own satisfaction, using three colours and a pair of scissors ! Later they made a set of solids each, and since their geometrical constructions are not yet very far advanced, they used squared paper, and cut the drawing down to a minimum. This is the only group with whom I propose to do much algebra, as I do not think that below the third stream much profitable work can be done in this subject.

Class 2.3 have done some very simple equations of the "think of a number" type, and next term we shall solve some equations using a balance and some pieces of cardboard. 2.3 have also worked out several ratios, preparatory to finding the heights of various objects by practical methods.

My aim throughout my teaching in this Comprehensive School has been to find the right type of Mathematics to suit each particular group of children, bearing in mind that the most profitable for at least half of the streams is the kind which they can put to use in their everyday life. This work uses up the time which could be devoted, as in past years, to a detailed, purely Arithmetical Syllabus, but I feel that more enjoyment, and a taste of the wonders revealed by Mathematics, should not be denied to even the dullest of our children.

Mr. M. W. Brown (Peckham) said that one of the main difficulties in the Comprehensive School was to find staff who were sufficiently interested in the subject to make it one of their special lines of teaching. Mathematics,

as far as he could see, was at present the Cinderella of subjects in the Comprehensive School. In many cases it was a subject for anybody who was sufficiently competent to do straightforward arithmetic. No one specialised in it as they did in science, modern languages, or geography, but staff who specialised in one of these subjects not infrequently filled in their time by teaching a little mathematics, which in many cases was nothing more than arithmetic. With so many people teaching mathematics—the staff of the school at which he worked numbered 52, and he imagined that 30 of these had something to do with the subject—it was very difficult indeed to get a clear policy in teaching.

He agreed entirely with what Mr. Swan had to say about the common core. Mathematics in their type of school required two or perhaps three syllabuses. The man who had to frame these syllabuses had a very large part indeed to play. It was likely to be a more difficult task, in his opinion, than that which faced the senior mathematical master of a normal grammar school. There were also administrative difficulties which had their repercussions on the teaching of the subject. In his own case a school of 1,200 boys was accommodated in two separate buildings about half a mile apart. In his building he had the young boys of all types, those who corresponded to the old central stream and those who corresponded to the old "senior" school. The older boys were in another building. If the senior mathematical master was really to supervise the teaching of mathematics, he should be in close and daily contact with the work in both buildings. It was difficult to move staff from one building to another. It had been tried and found not satisfactory. The staff certainly disliked it intensely. They had now reverted to the system in which a man remained in his own particular building, which meant that the senior mathematical master would be, not in his (the speaker's) building, but in the other. In pursuing a mathematical policy the master responsible should know the staff in both buildings, which in the circumstances described was difficult.

So far as the contents of the syllabus were concerned, he would mention only the question of geometry. Practical geometry ran like a canker throughout the teaching in many secondary modern and central schools. The course often consists of little more than learning a variety of constructions, with little of the framework of Stage A geometry or attempt to teach something of space comprehension. Its main purpose appears to be to serve as a preparation for Technical Drawing.

He desired to take up the point which Miss Giuseppi made about the "E" children. Mr. Swan must not suppose that he had got all the "E" children; the speaker had some of them. He kept the form as small as he could, but even so he had 33 children in the first year who were all either of the "E" category or suffered from some disability, either physical or mental. He was very uncertain at present how much mathematics should be taught to these children. Their great disadvantage was lack of verbal ability. They were usually unable to read, and a book was not of much use to them unless they could at least read very simple statements. Ideally these children had to be treated as individuals, and that being so, he did not think that books could be written to fit all the difficult cases that were received.

One other thing. Mr. Swan had spoken of starting with a bias towards the commercial or technical side. The speaker felt personally that that was a little disturbing. Many of these children were late developers, and their number was larger than he had expected. He found something like 15 or 20 boys of 13 + who at the end of the summer term were certainly good enough to take a grammar school course, but had developed late. He was successful in obtaining school grammar places for only a few of these children. In view

of this late development he would himself prefer that the boys should follow a course without bias until the end of their third year, when it would be easier to decide whether their work should be given a vocational bias or conform to the normal syllabus.

Miss Levin (Walworth) said that what had been brought forward by the openers corresponded to her own experience. One point of difference was the question of a common core ; in her school they believed that there should be a basic skeleton syllabus that should be covered by all streams, in perhaps a year and a half by the first stream and in three or more years by the slowest. The making of this syllabus was difficult, but they had tried to solve the problem by regular discussion meetings of all the teachers of mathematics in the school ; during the first term the " skeleton " had been worked out, and later the meetings had discussed methods and results.

Miss Levin wished that Mr. Swan's suggestion of half a day off for mathematics teachers for conferences could be adopted. A large stumbling-block had been the conservative attitude of some of her colleagues.

There must be one or two specialists in mathematics in each school, but she was not sure whether more were necessary.

Walworth had worked out a syllabus for what they called mathematical experience such as they felt important, and indeed essential for the children, and for the moment they were assuming that the children had very little experience when they entered the school. When the primary schools were freed from the syllabus which they were bound to follow at the moment, then the secondary school would have to review the situation.

With regard to " E " children, Miss Levin had been teaching two forms, and her present tentative feeling was that some special project which really interested them and would lead to verbal ability and the necessity of using practical mathematics might be the best approach.

At Walworth they were doing a certain amount of experimental work in the fourth year in connection with specialisation. Some forms had only girls, or only boys. Mathematics linked to technical or commercial courses was far too limited, and another approach was producing good results.

Miss Levin felt that it would be useful if mathematics teachers in the London comprehensive schools could meet together from time to time to exchange experiences. They might find that they were unknowingly doing similar things, or entirely different things, and would gain much by pooling their ideas.

Mr. K. B. Swaine (Yeovil) said that he felt a certain amount of diffidence in speaking on this subject because all his experience had been in grammar schools. But the subject was not without interest for him, because he felt there were always some in the ' B " forms of grammar schools who were not really capable of profiting by the academic mathematical course which was generally suitable. He very much envied Mr. Swan his satisfaction in the success of the selection of children at 11, because he doubted whether the distribution between the different sorts of schools was quite as good. He thought it impossible to draw anywhere a hard and fast line between grammar school and other school pupils. There must be some in each type of school who, in some subjects, were misfits, and he thought this overlap should be recognised and allowed for.

So far as teaching was concerned, one point struck him. He found boys only too ready merely to remember what they had been taught, and if only he could make them realise that something which they had thought of might be right, then he had succeeded. He put forward that idea as suitable for schools of all types.

Mr. C. T. Daltry (Institute of Education) said that in his view the pooling

of experience was the only profitable line of approach, and there was a great deal of experience available already, because children were taught in what were called senior elementary schools for a number of years. There was a certain amount of experience available with regard to the education of the ordinary boy, and it would be practicable for teachers working in these schools to meet and put their heads together and say what items were useful. They must try to get away from feelings of inferiority and realise what was required as essential. He suggested certain practical methods of approach to the subject—a discussion on how to find one's way about London, using the information furnished by the Transport Board's maps. Why not work on that basis instead of talking about the drawing of right angles? Why not a simple course for the making of models and toys? He could not imagine any normal human boy failing to respond to the proposition that someone should show him how to make a model aircraft or a model boat, and anybody could extract from this an enormous amount of mathematics. He offered these suggestions in the hope that something might come of them. He believed that the London Branch of the Association was concerned to get together the people interested, and if they could be got together they should be assisted by every possible resource.

Mr. A. W. Riley (Wolverhampton) supported the views advanced by the two opening speakers. As a mere provincial he confessed to one small disappointment. In the provinces they had thought that London really had comprehensive schools, and they found now that these were merely modern schools which overlapped the grammar school field a little more than usual. However, Mr. Swan had used the words " comprehensive " and " modern " as interchangeable. They were concerned with the education in mathematics of ordinary children, and in that connection it was necessary to re-examine the use of the term " common core ". Miss Giuseppi had outlined her syllabus for the fifth and sixth stream of a fully comprehensive school—that is to say, a stream about half-way down the whole age-group of children. She instanced many topics which were commonly included and to which no one would take exception. But a common core, strictly speaking, must be *common* to the top " A " of the grammar school and the bottom " F " or " G " of the educationally abnormal school. It must be part of the work of children who, to take Miss Giuseppi's example, had difficulty in reckoning 25 minutes after 11.25. A true common core could include only such mathematics as was learned by the normal child by the end of the primary stage. Too often the common core was common only to the grammar school and the top stream of the modern school, aiming deliberately at the transfer of one or two children from the modern school to the grammar school at 13+.

There were two other points he desired to underline.

One was the position of the mathematical teacher in these schools. Mr. Swan had mentioned that only three of his staff of 54 took mathematics as a main subject. Heads were painfully aware of the difficulty of getting people who would take mathematics as a main subject. General subjects teachers often thought mathematics too difficult for them. His own experience had been that the best men and women who came in under the Emergency Training Scheme, perhaps because they had no preconceived ideas, were often achieving notable success, although they were rarely mathematicians who could aspire to senior work in a grammar school.

The schools were already alive to the need for " group specialisation ", which, however, was only a halfway house between the ordinary situations of life and the accurate classification of the many subjects of the university. The development of group specialisation did raise one important point for members of the Mathematical Association. He knew quite a number of young

men and women who were keen to take up mathematics as a semi-specialist study, but their work often involved an additional demand for semi-specialisation in science or geography, or some other subject. The Association could hardly hope to get many of these people as full members; they would have equal reason for joining the Royal Geographical Society or the Science Masters' Association; the Association was not likely to see much of them except as Associate Members. He would commend to the Branches the need for roping them in in that category and doing the utmost they could for them, not regarding them as "poorer brethren", but recognising that their interests were much wider than those of mathematicians pure and simple. He had had some experience of interesting work with "emergency-trained" teachers during their probationary period, organising them in study groups on a group-specialist basis, including mathematics, science, and handicraft. In that group mathematics was the "common core", but one of the difficulties about the common core of mathematics was that it was so deep down, so very fundamental, that its presence often went unsuspected.

A final point concerned textbooks. Miss Giuseppi had mentioned how interesting it was to be dealing with a transitory stage. It was transitory in one sense—in the child's development year by year—but he hoped it would not be transitory in the development of education as a whole. She had pointed out, rightly, that when one got a new form of children each year the needs of those children were not the same as the needs of the corresponding form a year before. The schools were constantly facing new problems. Unfortunately, the moment somebody did a new job and found it worked well, somebody else came along and said, "Let us put it in a textbook". Even now they were getting a trickle of textbooks for secondary modern schools; if that trickle ever became a flood it would be a disaster. Secondary modern schools must base their work on the experience of the children, and this could not be confined within the covers of a book. He hoped the development of textbooks for the ordinary children would not be speeded up.

Mr. J. H. Burdon (Inspectorate, London) said that his contribution to the discussion was not intended to deal with any particular aspect of this work, but to follow upon what Mr. Daltry had just said. His work took him into a large number of secondary modern schools in London, and he had found, and so had his colleagues, that, particularly in mathematics, many teachers were more or less "floundering about". He did not use that expression in any derogatory sense at all, but merely in the sense that they did not know quite what to do, how to do it, and where to start. He remembered a discussion in the London Branch in the early part of 1949 when he called upon those in the audience who were actually doing this work to show their hands, and only two hands went up; in other words, in the London Branch it was realised that they were not doing anything whatever towards helping these people. As a result, a small *ad hoc* committee was formed to explore the situation and see what they could do about getting contact with the people teaching in the schools. About the middle of the year a note was put in the *Bulletin* inviting all interested teachers in London to write in. The same thing was done in Middlesex and Surrey. Seventy replies were received, and at the first meeting held last October it was astonishing what enthusiasm was shown amongst these people. It had been found in subsequent meetings, at which the average attendance was not less than fifty, that by far the best way of discussing these points was not to have a formal paper, but to get together, and after opening up some particular aspect, to break up into small groups of ten or so, have group discussions, and then all come together again, each group having appointed a spokesman, for general discussion. There are a number of people who are afraid to get up and talk in a large body, but are

not afraid to talk in a small group. He hoped that the London Branch would go ahead with these discussions. He felt that there was a tremendous job in front of the Branches of the Mathematical Association in getting these people together in that way.

Miss Giuseppi, in a brief reply, said that she had not meant it to be inferred that all classes should be fitted out with textbooks, and she would hate to be given one and told that that was the syllabus for the year ; but it was useful to have one or two textbooks for reference. She fully agreed that with regard to those in the eighth stream, most of the work was individual. There must be smaller classes with much more individual work.

She did not wholly agree with all Miss Levin's remarks, particularly when she said that they should ignore the primary school ; it would be very difficult, indeed impossible, to do so. Nor did she agree with Miss Levin's basic syllabus, which presupposed that one already knew the requirements of a group of children before they reached the school. She would not go as far as to have a rigid syllabus at any time. She saw no reason why they should bind themselves to one, since they had no need to satisfy any particular class of examination. She thought the syllabus should be as free as possible and adapted for each group.

Certainly it was much more interesting to talk about toys than to talk about right angles. When she mentioned the latter by name, she had in mind her brighter group.

Mr. F. J. Swan said that he felt he was expressing Miss Giuseppi's feelings as well as his own when he acknowledged the tenderness with which the audience had dealt with them. Of the intake of about 290 he managed to interview 240 of the pupils and their parents, which gave something of a flying start on the question of grading. He knew that there were difficulties, and he hoped they would be establishing contact with the primary schools and get a better first grade.

With regard to the staff moving about—a point mentioned by Mr. M. W. Brown—they had started from the first at West Norwood to move about. He had not heard any adverse comment from the staff, though such might have been made, of course, in the staff rooms. Mr. Brown, however, had to deal only with men, and men were notably obstinate ; his own was a mixed staff.

With regard to mechanical drawing, he loathed that subject. He remembered seeing in one school a picture which said, " We don't do Euclid here," and then there was a card which read, " To bisect a given straight line," followed by the usual diagram, and " Let AB be the given straight line," and so it went on. That was the sort of thing he deplored, and really he thought it ought to be forbidden in any school, let alone a comprehensive school.

He knew that general illiteracy—the inability to read—was a handicap when it came to dealing with mathematical problems, but he recalled one boy in the fifth stream of the fourth year who could not or would not read a word of three letters. Whenever this boy saw him and had any suspicion that he wanted to get on to him about reading, he quickly disappeared. Yet this boy was probably the best in the fourth year at practical mathematics. Perhaps they might have to think very seriously about their own symbolism—their own mathematical language. There was always the possibility that a boy might understand a mathematical language and yet fail to understand the English language. In fact, that was evident in some advanced textbooks of mathematics !

He wanted to straighten out one point on the question of bias in the third year. That bias was only in their very bright streams. There would be no

bias beyond the third and fourth streams of the seven. The boys there would certainly be doing the unflavoured mathematics he had mentioned.

He could not conceive that Miss Levin's definition of the common core was very useful. It was possible for a very dull child to achieve in twenty years what a mathematical expert would achieve in one, but that was not a common core.

One word about project working. At the emergency training colleges a great deal of work was done on projects, and he had the opportunity of seeing many of the experiments. He was not sure that they were suitable for all children. He quoted two cases which had come to his notice recently. One was of a project which extended over the whole term. One of the dull boys was asked, " What is your contribution to this? " His task was to place a flag on the local map at the position at which his house was situated. Asked if he had done anything else, he proudly said, " No." Even under the old-fashioned conditions he hesitated to think that the boy could have done much less.

Some of those present might not have heard the story of the Post Office project. A boy appeared at his home at the end of the morning session and said to his mother that he did not feel well, and he wanted to stay at home that afternoon. He was asked if he had got any pain, and replied, " Not particularly." He did not want to go to school. She asked him what he had been doing at the school, and he replied, " On Monday morning we were all given maps of the neighbourhood, and told to put red blobs on the positions where there were post offices and red crosses where there were pillar boxes. In the afternoon we had lessons about the post office. On Tuesday morning we had to draw post offices. On Tuesday afternoon we had sums about postal orders and stamps, and this morning we have been making models of post offices. I am fed up with the blasted post office ! " That was an extreme story, but it illustrated the need for care.

As for Mr. Swayne's difficulty of drawing a line and allowing for overlap, that overlap could be allowed for in one way only. There was no way other than the comprehensive scheme to ensure that all the pupils were in the right place. He thought it just came back to what he had said earlier. There was a desire to leave nobody outside the grammar school who could profitably be in, and there was neglect on the other side of the question because of those who were in the grammar schools and could not profit by the course. He could not see how there could be a two-way traffic to deal with that overlap.

With regard to Mr. Riley's comment on comprehensive and modern, he thought that if they had one more stream they would be fully comprehensive in range, provided, of course, they got rid of some of their " E "s.

He wished to pay a tribute to the work which Mr. Burdon had been doing for them in London. He had started conferences for teachers in mathematics, and he must be gratified when 70 or 80 gathered on Saturday afternoons to discuss the subject.

On the question of textbooks there was one difference between grammar schools and other schools. In the grammar school the publishers were allowed to show their wares ; in other schools this was not so. In his school they were allowed to go to Stockwell, if they had time, when the sample room was open there. A good deal could be done if the books were circulated in the schools.

He appreciated Mr. Burdon's remark that some of the teachers in the schools were fearful of talking in large meetings, but were ready to express themselves in small groups, which was an admirable plan.

"THE TEACHING OF MECHANICS IN SCHOOLS" AND OTHER REPORTS.

It is the duty of the Teaching Committee to keep under review Reports published by the Association, and accordingly members of this committee were asked last May to answer the following questions :

(*a*) Do you consider that the Mechanics Report is sufficiently in need of revision or rewriting for a sub-committee to be appointed now for this purpose, or do you consider that any other Report needs revision more urgently?

(*b*) Which parts of the Mechanics Report do you think are most in need of rewriting?

(*c*) Are there any topics not included in the Mechanics Report which you consider should be put into a new report?

Of the 25 members who replied, 6 were in favour of the appointment of a sub-committee now, 15 were in favour of revising another report first, and 4 were neutral.

Of the 15, 9 were quite definite, though some of them were prepared to allow slight revision of details ; the remaining 6 put forward various alternative suggestions. 7 of the 15 specified other reports which needed writing or revising before the Mechanics Report was touched, but their preferences were distributed over 5 different reports.

8 who replied were in favour of amplifying the Mechanics Report, and the greatest support was for the inclusion of a section on vectors.

These replies are analysed more thoroughly below. They have been considered by the Standing Sub-committee of the Teaching Committee, and they helped to indicate the course to be followed. There is a definite body of opinion that we should complete our reports on sixth form teaching by following up the existing Algebra Report with one on Higher Algebra and Trigonometry and Analysis for Sixth Forms. A proposal to this end is being put to the Sixth Form Sub-committee, which is also considering some observations collected in earlier years regarding the region common to the last year in the sixth form and the first year at the university. It seems likely that these topics will be dealt with first, and the Mechanics Report afterwards, though work on the latter may be begun by a different sub-committee while the work on the pure mathematics is still proceeding.

The Calculus Report was almost finished in 1949 and its distribution is planned for early in 1951. The Sixth Form Geometry Report and one on Visual Aids seem likely to make a close race of it for the honour of appearing next after that.

Most sub-committees this year have found that the change in membership caused by the appointment of a new Teaching Committee has led them to stop and reconsider their position and direction, but they have soon begun again with new vigour.

A new venture has been started by asking the Sixth Form and the Professional Training Sub-committees to combine in producing a short report on the place of the History of Mathematics in the teaching of mathematics.

We now turn to a further consideration of the replies to the questionnaire on the Mechanics Report. Most of the detailed suggestions made will be valuable when the Teaching Committee decides to appoint a sub-committee to consider revision of the Mechanics Report, whether by rewriting, amplifying or extending it. But in the absence of more agreement than was indicated above, little more can be done here than to mention some of the suggestions. A list of all of them would fill several pages of the *Gazette*; it seems more

profitable now to give a selection. It is true that the selection must be prejudiced by my own preferences, and I shall even venture to include some comments of my own, but such a course seems more likely to give rise to useful discussion. A sub-committee will have all the suggestions at its disposal, and one useful result of the questionnaire is that the replies indicate clearly who some of the members of such a sub-committee ought to be.

Here, then, are some of the views which our questions have elicited :—

The Report, as it stands, consists of 30 pages of introduction followed by 54 pages of odds and ends ; unification is needed.

A revised Report will have to take into account changes in syllabus which have resulted from the Jeffery Report, and also changes due to the introduction of the examination for the General Certificate of Education. The chapter on the earlier teaching of the subject needs changing appropriately. The Report should deal with Calculus and Vectors in their relations with Mechanics. More help is needed for those who take Dynamics before Statics in their teaching. Some guidance should be given on work for the end of the sixth form course and on such topics as the introduction of pupils in higher forms to theories of relativity and to quantum theory, with special reference to the relation between mass and energy.

One suggestion made was that the Report should be rewritten in three parts : an introduction (as a guide to the teacher) on the nature of the subject ; a " first course for all " ; and a second course for specialists, with clear guidance about its content.

The most important suggestion, to my mind, was that two stages should be recognised in the teaching of Mechanics, as in the teaching of Geometry—a stage A in which the lessons of experience are co-ordinated, and perhaps emphasised or extended by experiment, and a stage B of systematic deduction. It is in stage A that collaboration between Mathematics and Physics might be improved. In stage B pupils could be given a clearer view of the development of classical Applied Mathematics. For myself I would call the subject matter of these stages Mechanics and Applied Mathematics respectively ; using that terminology, I should say that much of the trouble in the teaching of Applied Mathematics arises from the tendency of pupils to go on to Applied Mathematics before they are sound in Mechanics. The existing Report stresses the advantage of appealing to experience when experiment is not available, or before beginning to experiment ; it does not so much stress the need for going quite a long way on fundamental principles with a minimum of mathematics before beginning to develop Applied Mathematics as a logical theory. Here we may note that one reply to the questionnaire cautioned against protracted discussions (whether in committee or on paper) about the relation of Applied Mathematics to the external world.

Three vigorously worded pronouncements among the replies ensure that a new sub-committee will not please everyone. It was suggested by one member that the revisers should settle once and for all the controversy about gravitational versus absolute units (on which, he said, teachers are almost equally divided) ; by another that they should come down definitely in favour of absolute units ; by the third that they be instructed not to re-argue all the old controversies.

This review of the situation of the Association's Reports generally, and of the Mechanics Report in particular, is being circulated to all members of the Teaching Committee, and at the same time is being sent to the Editor for publication in the *Gazette* for the information of all members of the Association.

August, 1950 J. T. Combridge.

MATHEMATICAL NOTES.

2151. *Corollaries to the " chord and tangent " theorem.*

1. *Pythagoras' theorem.*

In the figure, A is a right-angle, the circle on AC as diameter touches AB at A and also passes through D. Hence

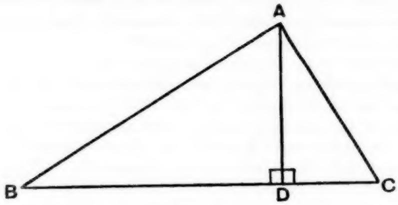


FIG. 1.

$$BD \,.\, BC = AB^2,$$

and similarly $$CD \,.\, BC = AC^2.$$

Adding, $$BC^2 = AB^2 + AC^2,$$

or $$a^2 = b^2 + c^2.$$

2. *Direct proof of the cosine formula.*

In Fig. 2, $ACDF$ and $ABDE$ are cyclic. Hence

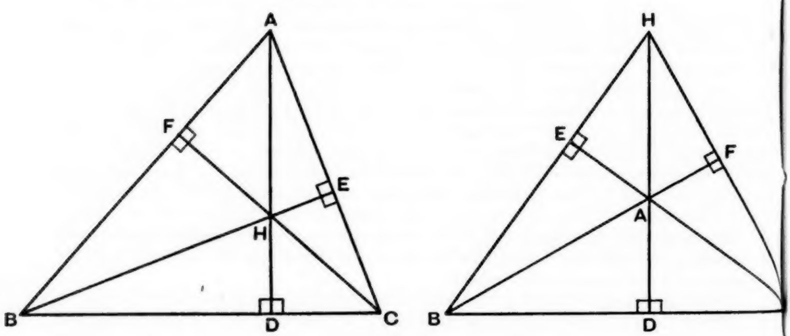


FIG. 2.

$$BD \,.\, BC = BF \,.\, BA,$$

$$DC \,.\, BC = CE \,.\, CA.$$

Adding, $$BC^2 = CE \,.\, CA + BF \,.\, BA.$$

But in each case, whether the triangle be acute-angled or not,

$$CE = b - c \cos A, \quad BF = c - b \cos A.$$

Thus $$a^2 = (b - c \cos A)\,b + (c - b \cos A)\,c,$$

or $$a^2 = b^2 + c^2 - 2bc \cos A.$$

3. *Pythagoras' theorem.*

In Fig. 3, the triangle ABC is right-angled at A, and the circle centre B and radius BA cuts CB at X and CB produced at Y. Then

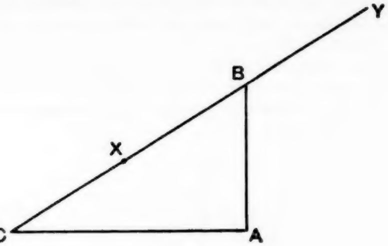


FIG. 3.

$$CX \,.\, CY = CA^2.$$

But $$CX = a - c,\ CY = a + c,\ CA = b.$$

Hence $$(a-c)(a+c) = b^2,$$

and so $$a^2 = b^2 + c^2.$$

4. *The cosine formula.*

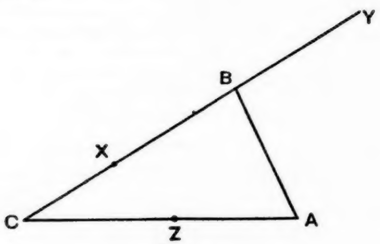


FIG. 4.

In the scalene triangle ABC $(a > c)$ let the circle centre B and radius BA cut CB at X, CB produced at Y, CA at Z. Then

$$CX \,.\, CY = CA \,.\, CZ.$$

Now $CX = a - c$, $CY = a + c$, $AZ = 2c \cos A$. Thus $CZ = b - 2c \cos A$. Therefore

$$(a-c)(a+c) = b(b - 2c \cos A),$$

or $$a^2 = b^2 + c^2 - 2bc \cos A.$$

The proof if $c > a$ involves a slightly different figure. If $\angle A$ is obtuse, A and Z are transposed and the proof proceeds as before. R. G. EVERITT.

2152. *Use of table of chords.*

Those who read in the *Report on the Teaching of Trigonometry* that the first trigonometrical table was a table of "chords", calculated by Hipparchus, may perhaps be interested to see how such a table can be used for the solution of plane triangles.

The "chord" of an angle θ is the length of a chord which in a circle of unit radius subtends an angle θ at the centre, so that

$$\text{chd } \theta = 2 \sin \tfrac{1}{2}\theta.$$

Hence the absence of a table of chords from the modern book of tables need not deter us from using them, or from considering whether our modern methods are so very much more shorter.

The table of chords was primarily for the isosceles triangle, and in using the table it is well to note that every triangle can be divided into four isosceles triangles by first dividing it into two right-angled triangles and then dividing each of these into two isosceles ones.

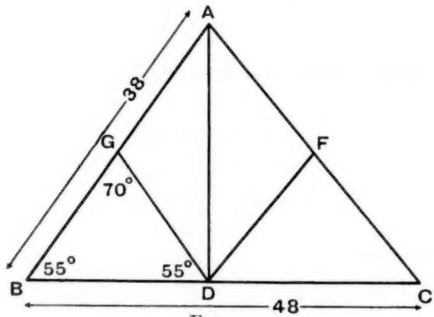


FIG.

Here is a possible solution from the data:

$$a = 48, \quad c = 38, \quad \angle B = 55°;$$

AD is perpendicular to BC; F, G are the mid-points of AC, AB, so that

$$GA = GB = GD = 19, \quad FA = FC = FD = z.$$

Then $BD = 19$ chd $70° = 21{\cdot}80$, so that $DC = 48 - 21{\cdot}80 = 26{\cdot}20$,

$$AD = 19 \text{ chd } 110° = 31{\cdot}13.$$

Hence, if $\angle AFD = \theta$ and $\angle DFC = \phi$, z chd $\theta = 31{\cdot}13$, z chd $\phi = 26{\cdot}20$; but $(\text{chd } \theta)^2 + (\text{chd } \phi)^2 = 4$, since

$$\sin^2 \tfrac{1}{2}\theta + \sin^2 \tfrac{1}{2}\phi = 1.$$

Thus $4z^2 = 31{\cdot}13^2 + 26{\cdot}20^2 = b^2$, whence $z = 20{\cdot}34$ and $b = 40{\cdot}68$.

Also $\quad \text{chd } \phi = 26{\cdot}20/20{\cdot}34 = 1{\cdot}2881,$

so that $\quad \phi = 80° \ 12',$

whence $\quad C = 49° \ 54'$ and $A = 75° \ 6'$.

In working out the above, use has been made of the table of squares and square-roots. It is not recorded that Hipparchus used these tables, but maybe he was sufficiently diligent to do without them. As his table of chords only went to the nearest degree, perhaps the answers should be given as $C = 50°$, $A = 75°$, $b = 40{\cdot}7$. C. O. T.

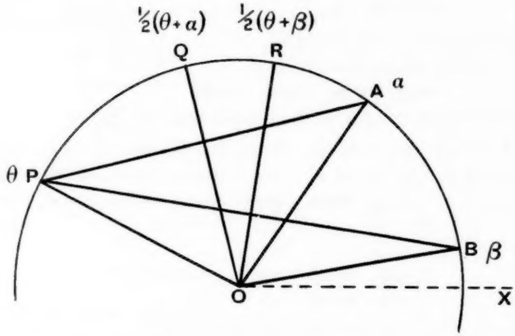
2153. *A neglected proof.*

We are, I think, apt to divide the properties of the circle into those which are obvious from symmetry, notably that concerning the perpendicular from

the centre to a chord, and the unsymmetrical angle properties which we all find so much more interesting.

The proof given below, which seems to be sadly neglected, shows, however, that the "angle" properties can be regarded as near-corollaries to the "symmetrical" properties.

Measuring angles round the circle in the usual way from an initial line OX, each point on the circle will be specified by an angle; thus in the figure, A is α where $\alpha = \angle AOX$, and similarly B is β and P is θ.



Now *by symmetry* the perpendicular bisectors of the chords AP and BP are radii, running from the centre O to the mid-points of the arcs AP, BP. If their ends are Q, R, then Q is the point $\frac{1}{2}(\theta+\alpha)$ and R the point $\frac{1}{2}(\theta+\beta)$.

Thus $\angle QOR = \frac{1}{2}(\theta+\alpha) - \frac{1}{2}(\theta+\beta) = \frac{1}{2}(\alpha-\beta)$.

But OQ and OR are at right angles to AP and BP respectively, whence

$$\angle APB = \tfrac{1}{2}(\alpha-\beta).$$

Thus if A, B are fixed and P varies, $\angle APB$ is constant and equal to half the angle subtended by AB at the centre. These are the first two of the "angle" properties.

If P lies on the minor arc between A and B, $\angle QOR$ is $\frac{1}{2}(\alpha-\beta)$ as before, but $\angle APB$ is the supplement of this angle. Hence the opposite angles of a cyclic quadrilateral are supplementary.

A circle could, I suppose, be defined as the plane figure with the maximum possible symmetry, so it is not unreasonable that even its apparently unsymmetrical properties should be deducible from considerations of symmetry.

C. O T.

2154. *The equation of a Pascal line.*

Challenged to verify Pascal's theorem in the crudest possible manner, Sir Charles Darwin sent me the coordinates of the point of intersection of two chords of an ellipse in a form from which the equation which Prof. Watson (vol. xxviii, p. 222) has traced back to the unidentified "D.O.S." follows at once. Taking the points with eccentric angles $2\theta_1$, ... on the ellipse, and writing

$$\theta_1 \pm \theta_4 = \phi_1, \psi_1; \quad \theta_3 \pm \theta_6 = \phi_2, \psi_2; \quad \theta_5 \pm \theta_2 = \phi_3, \psi_3,$$

we have for the intersection of the lines

$$(x/a)\cos(\theta_1+\theta_2) + (y/b)\sin(\theta_1+\theta_2) = \cos(\theta_1-\theta_2),$$
$$(x/a)\cos(\theta_4+\theta_5) + (y/b)\sin(\theta_4+\theta_5) = \cos(\theta_4-\theta_5),$$

the ratios

$$
\begin{aligned}
x/a : y/b : 1 \\
&= \sin(\theta_1+\theta_2)\cos(\theta_4-\theta_5)-\cos(\theta_1-\theta_2)\sin(\theta_4+\theta_5) \\
&\quad : \cos(\theta_1-\theta_2)\cos(\theta_4+\theta_5)-\cos(\theta_1+\theta_2)\cos(\theta_4-\theta_5) \\
&\quad\quad : \sin(\theta_1+\theta_2-\theta_4-\theta_5) \\
&= \sin(\phi_1-\psi_3)+\sin(\phi_3+\psi_1)-\sin(\phi_1+\psi_3)-\sin(\phi_3-\psi_1) \\
&\quad : \cos(\phi_1+\psi_3)+\cos(\phi_3-\psi_1)-\cos(\phi_1-\psi_3)-\cos(\phi_3+\psi_1) \\
&\quad\quad : 2\sin(\psi_1-\psi_3) \\
&= \sin\psi_3\cos\phi_1-\sin\psi_1\cos\phi_3 \\
&\quad : \sin\psi_3\sin\phi_1-\sin\psi_1\sin\phi_3 \\
&\quad\quad : \sin\psi_3\cos\psi_1-\sin\psi_1\cos\psi_3,
\end{aligned}
$$

and the line joining this point to the intersection (56, 23) is

$$
\begin{vmatrix}
x/a & \sin\psi_3\cos\phi_1-\sin\psi_1\cos\phi_3 & \sin\psi_3\cos\phi_2-\sin\psi_2\cos\phi_3 & \cos\phi_3 \\
y/b & \sin\psi_3\sin\phi_1-\sin\psi_1\sin\phi_3 & \sin\psi_3\sin\phi_2-\sin\psi_2\sin\phi_3 & \sin\phi_3 \\
1 & \sin\psi_3\cos\psi_1-\sin\psi_1\cos\psi_3 & \sin\psi_3\cos\psi_2-\sin\psi_2\cos\psi_3 & \cos\psi_3 \\
0 & 0 & 0 & \sin\psi_3
\end{vmatrix}=0,
$$

that is,

$$
\begin{vmatrix}
x/a & \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \\
y/b & \sin\phi_1 & \sin\phi_2 & \sin\phi_3 \\
1 & \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \\
0 & \sin\psi_1 & \sin\psi_2 & \sin\psi_3
\end{vmatrix}=0.
$$

The detail of interest is the use of the difference $\theta_5-\theta_2$ rather than $\theta_2-\theta_5$.

The corresponding equation for six points, $(a, 1/a)$ and so on, on $xy=1$ is deducible as

$$
\begin{vmatrix}
x & ad & cf & eb \\
y & 1 & 1 & 1 \\
1 & a & c & e \\
1 & d & f & b
\end{vmatrix}=0.
$$

This is the equation given in its expanded form by Watson (Note 1511, *Gazette*, xxv, p. 109; 1941). That this line does contain the intersection of the two lines

$$x+fay=f+a, \quad x+cdy=c+d$$

is obvious, for the determinant

$$
\begin{vmatrix}
f+a & ad+fa & cf+fa & eb+fa \\
c+d & ad+cd & cf+cd & eb+cd \\
1 & a & c & e \\
1 & d & f & b
\end{vmatrix},
$$

reduces at once to the nul form

$$
\begin{vmatrix}
0 & 0 & 0 & * \\
0 & 0 & 0 & * \\
1 & a & c & e \\
1 & d & f & b
\end{vmatrix}.
$$

But the effort of obtaining the determinantal form of the equation is very slight. The pair of equations

$$x+aby=a+b, \quad x+dey=d+e$$

gives

$$x : y : 1 = de(a+b) - ab(d+e) : (d+e) - (a+b) : de - ab$$
$$= ad(e-b) - eb(a-d) : (e-b) - (a-d) : de - ab\,;$$

and the pair of equations

$$x + bcy = b + c, \quad x + efy = e + f$$

gives

$$x : y : 1 = cf(e-b) - eb(c-f) : (e-b) - (c-f) : ef - bc.$$

Hence the Pascal line is

$$\begin{vmatrix} x & ad(e-b) - eb(a-d) & cf(e-b) - eb(c-f) & eb \\ y & (e-b) - (a-d) & (e-b) - (c-f) & 1 \\ 1 & de - ab & ef - bc & b \\ 0 & 0 & 0 & e-b \end{vmatrix} = 0,$$

reducing immediately to

$$\begin{vmatrix} x & ad & cf & eb \\ y & 1 & 1 & 1 \\ 1 & d & f & b \\ 0 & a-d & c-f & e-b \end{vmatrix} = 0,$$

and thence to the form required. E. H. N.

2155. *Sign Convention and Rotation.*

The reminder, in Note 2033, that in plane coordinate geometry and trigonometry the positive sense of a rotation is usually counterclockwise, leads us to seek the link between this custom and the convention that when the eye looks in the positive direction of an axis of rotation (for example an angular velocity vector) the rotation appears clockwise.

It will be seen that OX and OY, drawn respectively to the right and upwards, are really two of a right-handed set of three orthogonal axes, which is completed by directing OZ out of the paper *towards* the reader. The usual agreement is that a rotation is positive if it corresponds to a cyclical rotation of the axes. Thus a positive rotation about OZ turns OX towards OY, which is in accordance with the rule of plane geometry. When OX and OY are drawn in their usual directions, the rotation which turns OX on to OY, through the smaller angle, appears counterclockwise to the reader, although it is actually clockwise about the positive direction of OZ.

This is not always made clear at school, and it is only in later work that confusion results through mixing up right-handed and left-handed systems of axes.

The same rule of cyclical rotation would still be valid if the system were completed by directing OZ into the paper, but the modern trend seems to be towards right-handed axes and the regarding of clockwise rotations and couples as positive. For example, in the science of compass correction, a typical choice of axes fixed in a ship would be forward, starboard, and to-keel, or if fixed on the Earth, northward, eastward, and downward.

A. E. Williams.

2156. *A property of the quadrilateral and the construction of a quadrilateral to satisfy certain conditions.*

1. Let the sides AB, BC, CD, DA of a quadrilateral $ABCD$ be a, b, c, d.

Let the mid-points of these sides be P, Q, R, S, and let $QS=l$, $PR=m$. Then, if PR, QS meet at O, we have

$$OA^2+OB^2=\tfrac{1}{2}(a^2+m^2),\qquad OB^2+OC^2=\tfrac{1}{2}(b^2+l^2),$$
$$OC^2+OD^2=\tfrac{1}{2}(c^2+m^2),\qquad OD^2+OA^2=\tfrac{1}{2}(d^2+l^2).$$

Hence we see that

$$a^2-b^2+c^2-d^2=2(l^2-m^2),$$

a result which may be well known, but which I have seen only in *Gazette*, XXI, p. 51.

2. To construct the quadrilateral $ABCD$, given suitable values for a, b, c, d and l.

We first find m by using the result proved above. Then, taking O as origin and QOS as x-axis, we have the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$, $P(X, Y)$, $Q(-\tfrac{1}{2}, l)$, $R(-X, -Y)$, $S(\tfrac{1}{2}l, 0)$. We now have

$$(x_1-x_2)^2+(y_1-y_2)^2=a^2,\quad (x_2-x_3)^2+(y_2-y_3)^2=b^2, \qquad \text{(i), (ii)}$$
$$(x_3-x_4)^2+(y_3-y_4)^2=c^2,\quad (x_4-x_1)^2+(y_4-y_1)^2=d^2, \qquad \text{(iii), (iv)}$$
$$x_1+x_2=2X,\quad x_2+x_3=-l,\quad x_3+x_4=-2X,\quad x_4+x_1=l, \qquad \text{(v)}$$
$$y_1+y_2=2Y,\quad y_2+y_3=0,\quad y_3+y_4=-2Y,\quad y_4+y_1=0. \qquad \text{(vi)}$$

Equations (i)–(iv) may be written :

$$(2x_1-2X)^2+(2y_1-2Y)^2=a^2,\quad (-2x_1+4X+l)^2+(-2y_1+4Y)^2=b^2,$$
$$(2x_1-2X-2l)^2+(2y_1-2Y)^2=c^2,\quad (-2x_1+l)^2+(-2y_1)^2=d^2.$$

From these and equations (v) we obtain

$$8lx_1=8lX+4l^2+a^2-c^2,\qquad 8lx_2=8lX-4l^2-a^2+c^2,$$
$$8lx_3=-8lX-4l^2+a^2-c^2,\qquad 8lx_4=-8lX+4l^2-a^2+c^2. \qquad \text{(vii)}$$

Using also equations (vi) we obtain

$$16lYy_1=16lY^2-l(b^2-d^2)-2X(a^2-c^2),$$
$$16lYy_2=16lY^2+l(b^2-d^2)+2X(a^2-c^2),$$
$$16lYy_3=16lY^2-l(b^2-d^2)-2X(a^2-c^2),$$
$$16lYy_4=16lY^2+l(b^2-d^2)+2X(a^2-c^2). \qquad \text{(viii)}$$

If we now substitute for x_1, y_1, x_2, y_2 obtained from (vii) and (viii) into (i) we have

$$4Y^2\{16a^2l^2-(4l^2+a^2-c^2)^2\}=\{l(b^2-d^2)+2X(a^2-c^2)\}^2. \qquad \text{(ix)}$$

But $4X^2+4Y^2=m^2$, so that (ix) may be put into the form

$$32X^2l^2(a^2+c^2-2l^2)+4Xl(a^2-c^2)(b^2-d^2)+l^2(b^2-d^2)^2$$
$$+m^2(a^2-c^2)^2-8l^2m^2(a^2+c^2-2l^2)=0. \qquad \text{(x)}$$

Substituting similarly from (vii) and (viii) into (ii), (iii) and (iv) does not lead to results essentially different from (x).

The method is therefore to solve (x), which, when the data are suitably chosen, gives two real values for X, then to find the corresponding values of Y (the positive value may be taken) from (ix) or from the equation

$$4X^2+4Y^2=m^2,$$

and then to use equations (vii) and (viii) to determine the positions of A, B, C and D. Knowing X, Y, and therefore the positions of P, Q, R and S, we can, of course, construct the quadrilateral when we have found the position of A.

Equations (viii) cannot be used for finding y_1, y_2, y_3, y_4 in the special cases in which $Y=0$, but other equations may then be used.

Figs. 1–4 have been drawn using the following data :
Given $a=3$, $b=7$, $c=6$, $d=2$, $l=4$; then $m=\sqrt{20}$.

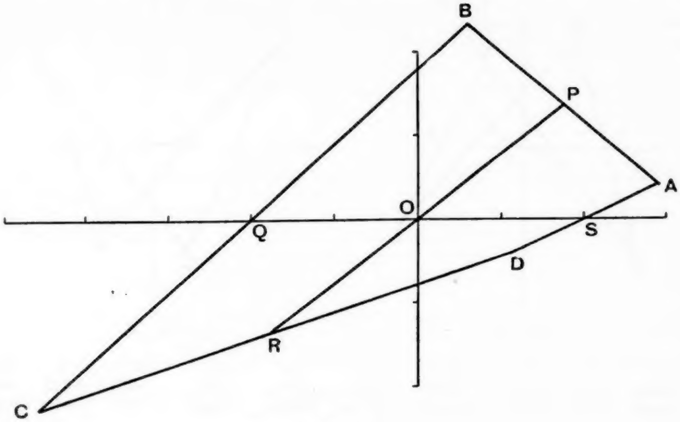


FIG. 1.

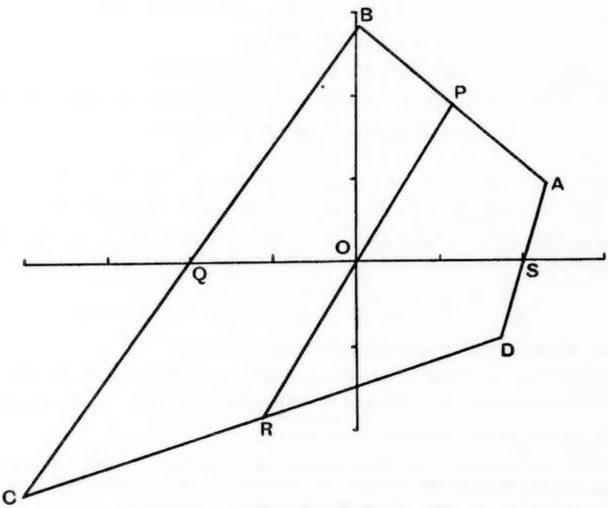


FIG. 2.

X	Y	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4
1·733	1·414	2·889	0·459	0·577	2·370	−4·577	−2·370	1·111	−0·459
1·188	1·894	2·344	0·938	0·032	2·849	−4·032	−2·849	1·656	−0·938

Given $a=\sqrt{10}$, $b=\sqrt{8}$, $c=3$, $d=\sqrt{5}$, $l=2$; then $m=1$.

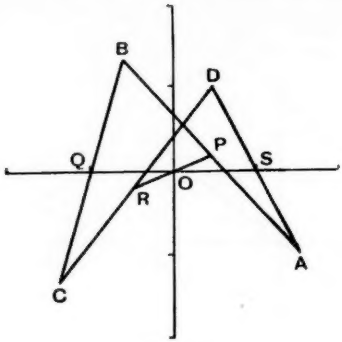


FIG. 3.

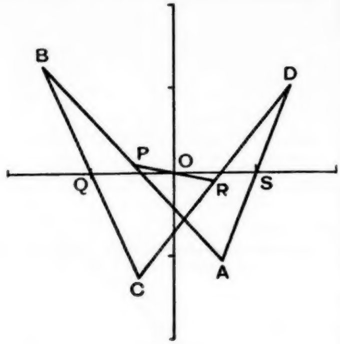


FIG. 4.

X	Y	x_1	y_1	x_2	y_2	x_3	y_3	x_4	y_4
0·465	0·184	1·527	−0·993	−0·598	1·361	−1·402	−1·361	0·473	0·990
−0·482	0·135	0·581	−1·035	−1·544	1·305	−0·456	−1·305	1·419	1·035

When the roots of equation (x) in X are equal, only one quadrilateral can be drawn. The condition for this is

$$(a^2-c^2)^2(b^2-d^2)^2=8(a^2+c^2-l^2)\{l^2(b^2-d^2)^2+m^2(a^2-c^2)^2$$
$$-8l^2m^2(a^2+c^2-2l^2)\},$$

which, using $a^2+c^2-2l^2=b^2+d^2-2m^2$, can be written

$$(a^2-c^2)^2(b^2-d^2)^2=8l^2(a^2+c^2-2l^2)(b^2-d^2)^2+8m^2(b^2+d^2-2m^2)(a^2-c^2)^2$$
$$-64l^2m^2(a^2+c^2-2l^2)(b^2+d^2-2m^2),$$

which is

$$\{8l^2(a^2+c^2-2l^2)-(a^2-c^2)^2\}\{8m^2(b^2+d^2-2m^2)-(b^2-d^2)^2\}=0$$

or $$\{4l^2-(a+c)^2\}\{4l^2-(a-c)^2\}\{4m^2-(b+d)^2\}\{4m^2-(b-d)^2\}=0.$$

Taking $2l=a+c$, we find that $X=-(b^2-d^2)/4(a-c)$, $y_1=y_2$, $y_3=y_4$, and the figure is a trapezium. Similar results follow by taking

$$2l=a-c\ (a>c),\quad 2l=c-a\ (c>a),$$
$$2m=b+d,\quad 2m=b-d\ (b>d),\quad 2m=d-b\ (d>b).$$

Also, the roots of (x) are real and unequal when

(i) $4l^2>(a+c)^2$ or $4l^2<(a-c)^2$, and $4m^2>(b+d)^2$ or $4m^2<(b-d)^2$,

or (ii) $(a+c)^2>4l^2>(a-c)^2$, and $(b+d)^2>4m^2>(b-d)^2$. H. J. CURNOW.

2157. *Sur certains polygones inscriptibles.*

Mathot a démontré un théorème et sa réciproque * qui nous ont servi récemment à propos des points de Feuerbach d'un triangle.† Voici une généralisation de ces deux propositions utiles à connaître.

THÉORÈME. *Soit* $(P)\equiv A_1A_2A_3\ldots A_{4n+2}$ *un polygone de* $4n+2$ *côtés* (*con-*

* *Mathesis*, 1897, p. 139.

† *Mathesis*, t. LV, p. 256.

vexe ou non) inscrit à une circonférence, dont on désigne par 1, 2, 3, ... *le rang des côtés successifs, et tel que les diagonales principales qui joignent les sommets opposés concourent en un même point O. Le produit des longueurs des côtés de rang impair est égal au produit des longueurs des côtés de rang pair.*

Pour abréger l'écriture, soient a_1, a_2, a_3, ... , a_{4n+2} les longueurs des côtés A_1A_2, A_2A_3, A_3A_4, ... , $A_{4n+2}A_1$ du polygone (P) considéré que parcourrait un mobile partant du sommet A_1 et allant vers A_2.

Les triangles semblables OA_1A_2 et $OA_{2n+3}A_{2n+2}$ donnent

$$\frac{a_1}{a_{2n+2}}=\frac{OA_1}{OA_{2n+2}}.$$

En considérant chaque système analogue de triangles semblables, on a la suite d'égalités

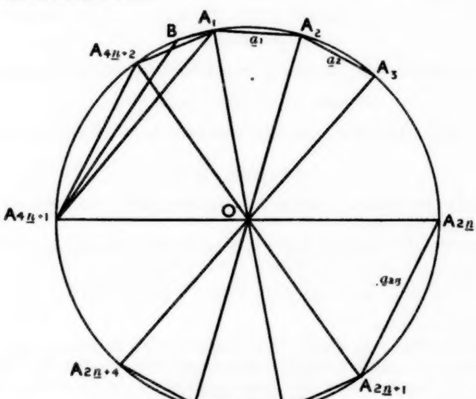
$$\frac{a_{2n+3}}{a_2}=\frac{OA_{2n+1}}{OA_2},\quad \frac{a_3}{a_{2n+4}}=\frac{OA_3}{OA_{2n+5}},\ldots,\ \frac{a_{4n+1}}{a_{2n}}=\frac{OAa_{4n+1}}{OA_{2n}},\quad \frac{a_{2n+1}}{a_{4n+2}}=\frac{OA_{2n+1}}{OA_1}.$$

Multipliant ces $2n+1$ égalités membre à membre, il vient

$$\frac{a_1 \,.\, a_3 \ldots a_{2n+1} \,.\, a_{2n+3} \ldots a_{4n+1}}{a_2 \,.\, a_4 \ldots a_{2n} \,.\, a_{2n+2} \ldots a_{4n+2}}=\frac{(OA_3 \,.\, OA_{2n+4}) \ldots (OA_{2n+1} \,.\, OA_{4n+2})}{(OA_2 \,.\, OA_{2n+3}) \ldots (OA_{2n} \,.\, OA_{4n+1})}=1, \quad \ldots.\text{(i)}$$

car, au second membre, les produits entre parenthèses sont égaux entre eux.

Réciproque Partielle.* *Si un polygone inscriptible convexe* $(P)\equiv A_1A_2 \ldots A_{4n+2}$ *de* $4n+2$ *côtés est tel que le produit des longueurs des* $2n+1$ *côtés de rang impair est égal au produit des longueurs des* $2n+1$ *côtés de rang pair et que* $2n$ *diagonales principales concourent en un même point O, les* $2n+1$ *diagonales principales sont concourantes.*



Car, si la droite qui joint le sommet A_{2n+1} au point d'intersection O des droites A_1A_{2n+2} et $A_{2n}A_{4n+1}$ rencontre la circonférence la seconde fois en un point B, on a, en vertu du théorème précédent,

$$A_1A_2 \,.\, A_3A_4 \,.\, \ldots \,.\, A_{4n+1}B=A_2A_3 \,.\, A_4A_5 \,.\, \ldots \,.\, BA_1.$$

* Le cas où (P) n'est pas convexe sera considéré dans une note qui paraîtra dans l'un des prochains fascicules de *Mathesis*.

En comparant cette égalité à l'hypothèse (i), on trouve

$$\frac{A_{4n+1}A_{4n+2}}{A_{4n+1}B}=\frac{A_{4n+2}A_1}{BA_1}.$$

Les triangles $A_1A_{4n+2}A_{4n+1}$ et A_1BA_{4n+1} ayant un angle égal en B compris entre deux côtés homologues proportionnels sont donc semblables et les angles $(A_1A_{4n+2}, A_1A_{4n+1})$ et (A_1B, A_1A_{4n+1}) sont égaux. Par suite les points A_{4n+2} et B se confondent et toutes les diagonales principales du polygone (P) concourent au point O.

N.B. Mathot a considéré le cas où $n=1$ (*loc. cit.*). V. Thébault.

2158. *On differentials.*

I have just read in No. 306 a paper by Mr. Phillips on differentials. The writer defines higher differentials by introducing the classic conception supposing the differential of the independent variable to be a constant.

Usually adopted as it is, it has ever been impossible for me to admit such a conception. My reason for that is that I cannot admit what I cannot understand, and this one is for me fully unintelligible, as it speaks of an infinitesimal which is a constant. I only know of one constant which is also infinitesimal, *viz.* the constant zero.

In my opinion, the notion of differential, as formulated by the founders of Infinitesimal Calculus, must be considered as obsolete. As Poincaré has already said, "We must think in derivatives."

$$dy=y'(x)\,dx$$

means that, if x and, consequently, y are expressed in terms of any auxiliary variable t, we have

$$dy/dt=y'(x)\,.\,dx/dt.$$

Similarly $$dz=p\,.\,dx+q\,.\,dy \qquad (1)$$

means that, whatever way x and y are functions of a variable u, we have

$$dz/du=p\,.\,dx/du+q\,.\,dy/du.$$

As long as only the first order is concerned, we could also say that (1) means that

$$\Delta z=p\Delta x+q\Delta y \qquad (2)$$

is approximately true when x and y are infinitesimals; and this definition could replace the preceding one.

But for the second order, we have nothing such (though some very artificial generalizations of (2) have sometimes been proposed). For me,

$$d^2(z)=p\,.\,d^2x+q\,.\,d^2y+r\,.\,dx^2+2s\,.\,dx\,dy+t\,.\,dy^2$$

means nothing else than

$$d^2z/du^2=p\,.\,d^2x/du^2+q\,.\,d^2y/du^2+r(dx/du)^2+2s\,.\,dx/du\,.\,dy/du+t(dy/du)^2$$

when x and y are any (differentiable) functions of a variable u. The advantage of cancelling the denominator du^2 corresponds to the fact that this auxiliary variable u and the functions $x(u)$ and $y(u)$ are arbitrary.

As to

$$d^2z=r\,dx^2+2s\,dx\,dy+t\,dy^2,$$

I consider it as meaningless.

The idea of "making the differential of the independent variable a constant" is all the more to be rejected as the only use of differential notation consists in leaving the independent variable fully indeterminate (as is essential when the problem is one of change of variables).

J. Hadamard.

2159. *Some Summation Formulae for Binomial Coefficients.*

In his review of H. S. M. Coxeter's *Regular Polytopes* (*Math. Gaz.*, vol. 33, p. 49) Mr. H. Martyn Cundy quotes the formula

$$\sum_{s=0} \left(-\tfrac{1}{4}\right)^s \binom{n-s}{s} = (n+1)/2^n,$$

and supplies the proof by considering the coefficients of x^n in the expansion of

$$\frac{1}{1-x}\left[1+\frac{\frac{1}{4}x^2}{1-x}\right]^{-1} = (1-\tfrac{1}{2}x)^{-2}.$$

Ingenious as this certainly is, it suffers in the present writer's mind from the disadvantage that it makes the result appear a curiosity rather than a particular case of a theorem of wider application. I venture therefore to give the following derivation of this and other results.

Let $\sum_{s=0} \binom{n-s}{s} t^s$ be denoted by P_n, then we have

$$\Delta P_n = P_{n+1} - P_n = \sum_{s=0} \binom{n-s}{s-1} t^s \quad \text{and} \quad \Delta^2 P_n = \Delta P_{n+1} - \Delta P_n = \sum_{s=0} \binom{n-s}{s-2} t^s.$$

Introducing $s-1=u$ and remembering that ${}^mC_k$ is zero when k is negative, we have

$$\Delta P_n = \sum_{u=0} \binom{n-u-1}{u} t^{u+1},$$

and

$$\Delta^2 P_n = \sum_{u=0} \binom{n-u-1}{u-1} t^{u+1}.$$

Hence

$$\Delta^2 P_n + \Delta P_n = \sum_{u=0} \binom{n-u}{u} t^{u+1} = tP_n.$$

This is a difference equation for P_n which may be written

$$P_{n+2} - P_{n+1} - tP_n = 0.$$

The roots of its characteristic equation, *viz.* $x^2 - x - t = 0$ are

$$\tfrac{1}{2}[1+(1+4t)^{\frac{1}{2}}] \quad \text{and} \quad \tfrac{1}{2}[1-(1+4t)^{\frac{1}{2}}],$$

and thus

$$P_n = \frac{\alpha}{2^n}[1+(1+4t)^{\frac{1}{2}}]^n + \frac{\beta}{2^n}[1-(1+4t)^{\frac{1}{2}}]^n,$$

with α and β independent of n. They are found, from $P_0 = P_1 = 1$, to be

$$\alpha = [1+(1+4t)^{\frac{1}{2}}]/2(1+4t)^{\frac{1}{2}} \quad \text{and} \quad \beta = [-1+(1+4t)^{\frac{1}{2}}]/2(1+4t)^{\frac{1}{2}}$$

which gives finally

$$P_n = \frac{1}{(1+4t)^{\frac{1}{2}}}\left[\left(\frac{1+(1+4t)^{\frac{1}{2}}}{2}\right)^{n+1} - \left(\frac{1-(1+4t)^{\frac{1}{2}}}{2}\right)^{n+1}\right].$$

For the sake of illustration, take

$$t=1\,; \quad \sum_{s=0}\binom{n-s}{s} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right],$$

$$t=2\,; \quad \sum_{s=0} 2^s \binom{n-s}{s} = \tfrac{1}{3}(2^{n+1}+(-1)^n).$$

A more interesting case arises from $t=-1$. Since

$$\tfrac{1}{2}(1+\sqrt{-3}) = e^{i\pi/3} \quad \text{and} \quad \tfrac{1}{2}(1-\sqrt{-3}) = e^{-i\pi/3},$$

we obtain for

$$t=-1\,;\ \sum_{s=0}(-1)^s\binom{n-s}{s}=[e^{i(n+1)\pi/3}-e^{-i(n+1)\pi/3}]/i\sqrt{3}$$
$$=\frac{2\sin(n+1)\pi/3}{\sqrt{3}},$$

which is $=0$ for $n=3m+2$ and $=(-1)^m$ for $n=3m$ or $n=3m+1$.

The case $t=-\frac{1}{4}$, from which we started, leads first to the indefinite form 0/0, but we can expand the expression for P_n, and obtain

$$\frac{1}{2^{n+1}(1+4t)^{\frac{1}{2}}}[1+(n+1)(1+4t)^{\frac{1}{2}}+\ldots-1+(n+1)(1+4t)^{\frac{1}{2}}-\ldots]$$

$=(n+1)/2^n+$ terms which disappear for $1+4t=0$, which is the result we wished to obtain. Alternatively, we could go back to the difference equation which, for $t=-\frac{1}{4}$, reads $P_{n+2}-P_{n+1}+\frac{1}{4}P_n=0$. The characteristic equation has now a double root $\frac{1}{2}$ and the solution of the difference equation is therefore $P_n=\alpha(\frac{1}{2})^n+\beta n(\frac{1}{2})^n$. $P_0=P_1=1$ gives at once $\alpha=\beta=1$ and hence $P_n=(n+1)/2^n$, as before. S. VAJDA.

2160. *The fundamental theorem of arithmetic.*

A proof of the fundamental theorem of arithmetic which depends on showing that any number divisible by p_1^α, p_2^β, … , p_k^κ, where p_i are distinct primes and α, β, … , κ are integers, is divisible by the product $p_1^\alpha p_2^\beta \ldots p_k^\kappa$.

This theorem, which is the basis of systematic arithmetic, has been proved as a deduction from Euclid's Theorem (that if p is prime and $p \mid m_1 . m_2$ then $p \mid m_1$ or $p \mid m_2$) and more recently by the method of induction.* In the present proof, the uniqueness of the representation of a number as a product of primes is shown directly, the simple theorem that any integer *is* a product of primes being proved incidentally.

Theorem 1A. The least number divisible by m_1, m_2, which have no common divisor, is the product $m_1 . m_2$.

Let N be the least number divisible by m_1, m_2.

Then $$m_1m_2 \geqslant N.$$

Now the sequence m_1m_2-N, m_1m_2-2N, … , m_1m_2-kN, … ultimately contains negative terms. If $m_1m_2-\theta N$ is the least non-negative member of the sequence

$$N>m_1m_2-\theta N \geqslant 0.$$

Since $m_1m_2-\theta N$ is divisible by m_1, m_2, and N is the least such number, we cannot have

$$N>m_1m_2-\theta N>0.$$

Therefore $m_1m_2=\theta N$; and since $N=\lambda_1 m_1=\lambda_2 m_2$,

$$m_1=\theta\lambda_2, \quad m_2=\theta\lambda_1,$$
$$\theta=1,$$

since m_1, m_2 have no common divisor.

i.e. $$m_1m_2=N.$$

Theorem 1B. Any number divisible by m_1, m_2, which have no common divisor, is divisible by the product $m_1 . m_2$.

$$m_1, m_2 \mid m_1 . m_2 + r \rightarrow m_1, m_2 \mid r$$

* See *e.g.* F. A. Lindemann, *Quarterly Journal of Math.* (Oxford), 4 (1933), 319–20.

and so the least number greater than $m_1 . m_2$ which is divisible by m_1, m_2 is $2m_1 . m_2$. It is clear by repeating this argument that any number divisible by m_1, m_2 is of the form $\mu m_1 . m_2$, and the theorem follows.

Theorem 2. Any number divisible by $m_1, m_2, m_3, \ldots, m_n$, no two of which have a common divisor, is divisible by the product $m_1 . m_2 . m_3 \ldots m_n$.

This theorem follows by induction from Theorem 1. For if the statement of Theorem 2 is true for $(n-1)$ numbers m_i, then any number M divisible by $m_1, m_2, \ldots, m_n$ is divisible by the products $m_2 . m_3 \ldots m_n, m_1 . m_3 \ldots m_n$.

i.e.
$$M = \lambda_1 m_2 m_3 \ldots m_n = m_1 \lambda_2 m_3 \ldots m_n ;$$
$$\lambda_1 m_2 = m_1 \lambda_2 = \lambda m_1 m_2 \quad \text{by Theorem 1B.}$$
$$\lambda_1 = \lambda m_1, \quad \lambda_2 = \lambda m_2 ;$$
$$M = \lambda m_1 . m_2 . m_3 \ldots m_n.$$

Since the theorem is true for $n = 2$ (Theorem 1), by induction it is true for all n.

Theorem 3. If p is prime, p^α is not divisible by any prime q distinct from p.

For otherwise p^α is divisible by the product $p . q$ (Theorem 1B) and so $p^{\alpha-1}$ is divisible by q; by repeating this argument p is divisible by q, which is absurd.

Corollary. If p_1, p_2 are two distinct primes, p_1^α, p_2^β have no common divisor.

Fundamental Theorem. If $p_1, p_2, \ldots, p_k$ are the prime divisors of a number n, with multiplicities $\alpha, \beta, \ldots, \kappa$, then
$$n = p_1^\alpha . p_2^\beta \ldots p_k^\kappa,$$
and apart from rearrangement of factors, this expression for n as a product of primes is unique.

No two of the numbers $p_1^\alpha, p_2^\beta, \ldots, p_k^\kappa$ have a common divisor (by the corollary to Theorem 3) and so by Theorem 2
$$n = \mu p_1^\alpha . p_2^\beta \ldots p_k^\kappa.$$
Now $\mu = 1$, for otherwise n would be divisible by a prime distinct from p_i, or by a higher power of one of the p_i's. This expression is unique since n has no other prime divisors.

O. G. JONES.

2161. *On Note* 2018.

Mr. Robson's Note 2018 (*Gazette*, No. 300, July, 1948) contains much with which I should agree if I accepted his convention; it also contains one red herring in the introduction of the fallacy Secundum Quid.

I agree that it seems inconsistent with our original definitions to use the radical in surdic equations and integrals as if it were unambiguous, and I dislike the statement now commonly made in textbooks that cis $\dfrac{p\theta}{q}$ is one of the values of $(\text{cis}\ \theta)^{p/q}$, (Chrystal never falls into this trap). But it is our custom sometimes to relax at a later stage the strict controls we impose on our pupils at the most elementary stage. Mr. Durell's textbooks emphasise that letters must be used for numbers, not for numbers-of-things, but he has no hesitation in writing such an equation as $\sin x = 0{\cdot}5$.

If, however, I adopted Mr. Robson's convention I should find some difficulty in explaining to my beginners that in this kind of algebra multiplication is not commutative: for with the convention $(4^{\frac{1}{2}})^2$ is certainly positive while $(4^2)^{\frac{1}{2}}$ is ambiguous, and the third law of indices does not necessarily hold.

B. A. SWINDEN.

2162. *Something beginning with a " B ".*

For my sins I have had recently to compile a card index for a small Mathematical Library, and it surprised me greatly to see how many mathematicians who write books have names beginning with a " B ". Possibly " B " is the commonest initial for all English names; it certainly seems to be so with mathematicians. Not only are there more names but many of them seem to write several books.

How these names bring back school and university days! There was good old Borchardt and Perrott's *Trigonometry* for instance, which must have been used by thousands in its time. Bell's *Coordinate Geometry of Three Dimensions*, Barlow and Bryan's *Astronomy* and Burnside and Panton's *Theory of Equations* are three good old stagers, the latter being for me a stand-by often referred to. Baker is there of course, and Bromwich and Brodetsky, and of course Barlow of table fame. What should we do without Barlow's lovely tables? It is a joy just to turn over the pages.

It is pleasant to see Basset, Besant and Boole, three names so well-known to old-timers. I don't suppose they are read much now. Briggs and Bryan swell the list—there are ten cards with their names on them; then there is Barnard covering much the same field. There are eight cards with the name Borel, namely the first eight of the Borel tracts—the ninth is by Baire. Bieberbach has eight cards too; I wish I could say that books of his had been used a lot, but knowledge of German is not a strong point in a far-flung Colony. Rouse Ball is there of course and very popular, and the list has been swelled of late by the American E. T. Bell.

What a feast of well-known names!

I must finish the list with an important " author " whose name begins with a " B ". This is the British Association, whose tables were more numerous before the Japanese got their hands on them. They only left us two, and so far we have only been able to get one replacement for the others.

Would anybody hazard a guess as to which comes next in the list? Actually it is " S ", closely followed by " L ", the latter of course being greatly swelled by Lamb and Loney. Both of these letters are, however, a long way behind the " B "s. J. C. COOKE (Singapore).

2163. *Inequalities in the triangle.*

In Note 2027 of No. 300 of the *Mathematical Gazette*, Professor Krishnaswami Ayyangar shows a few simple ineqalities for the triangle, based on the theorem of Means. His examples are concerned with angles, so that it may be of interest to add a few more inequalities involving lengths.

Thus from

$$\tfrac{1}{3}\{(s-a)+(s-b)+(s-c)\} \geqslant \sqrt[3]{\{(s-a)(s-b)(s-c)\}}$$

we deduce immediately

$$s \geqslant 3\sqrt{3}r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again from

$$\tfrac{1}{3}(a+b+c) \geqslant \sqrt[3]{(abc)},$$

we deduce $\quad 8s^3 \geqslant 27 \,.\, 4R\Delta,$

whence $\quad 2s^2 \geqslant 27Rr. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Starting from $\quad \tfrac{1}{2}\{(s-a)+(s-b)\} \geqslant \sqrt{\{(s-a)(s-b)\}}$

we have $\quad c \geqslant 2\sqrt{\{(s-a)(s-b)\}},$

so that by multiplication from this and the two other similar inequalities,

$$abc \geqslant 8(s-a)(s-b)(s-c),$$

which gives $\quad R \geqslant 2r. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$

Using (3) and the well-known equality $r_1+r_2+r_3-r=4R$ we have at once

$$r_1+r_2+r_3 \geqslant 9r. \qquad \text{(4)}$$

A rather longer piece of calculation may be effected from the same inequality

$$\tfrac{1}{2}\{(s-a)+(s-b)\} \geqslant \surd\{(s-a)(s-b)\}.$$

We have, as above, $\qquad c \geqslant 2\surd\{(s-a)(s-b)\}$

so that $\qquad c \geqslant 2\surd(ab) \,.\, \sin \tfrac{1}{2}C.$

Replacing c by $2R \sin C$ and using the formula for $\cos \tfrac{1}{2}C$ we have

$$2R\surd\{s(s-c)\} \geqslant ab.$$

Now $\tfrac{1}{2}\{s+(s-c)\} > \surd\{s(s-c)\}$, so that $R(a+b) > ab$, that is

$$1/R < 1/a + 1/b.$$

By adding the two similar results,

$$3/2R < 1/a + 1/b + 1/c. \qquad \text{(5)}$$

Another interesting approach to the question of inequalities in the triangle may be made by using any formula for the square of a length (necessarily positive) between two points which coincide in the case of the equilateral triangle.

Thus $OI^2 = R^2 - 2Rr$ gives immediately $R \geqslant 2r$, which is inequality (3) above ;

$$OH^2 = R^2 - 8R^2 \cos A \cos B \cos C$$

gives $\qquad \cos A \cos B \cos C \leqslant \tfrac{1}{8}. \qquad$ (6)

Again, if we describe an equilateral triangle XBC so that A and X lie on the same side of BC, it is easy to prove, using the cosine formula on the triangle AXB, that

$$AX^2 = a^2 + b^2 + c^2 - 4\surd 3 \Delta,$$

whence we deduce

$$a^2 + b^2 + c^2 \geqslant 4\surd 3 \Delta. \qquad \text{(7)}$$

As a final example of this general approach, I would mention the inequality

$$3R^2(\Sigma a^2 + 12\surd 3 \Delta) \geqslant 4\Delta\,(4\Delta + \surd 3 \Sigma a^2), \qquad \text{(8)}$$

which may be deduced by calculating the square of the distance of a Brocard point from the circumcentre.

Further results may of course be deduced by applying any of these inequalities to any associated triangle such as the ex-central triangle or the pedal triangle.

F. M. Goldner.

2164. *The catenary as a " simple " curve.*

It is well known that the length (s) of a curve, $f(x, y)$, referred to a set of rectangular coordinates, is expressed by the equation : $(ds)^2 = (dx)^2 + (dy)^2$. This equation can be put in the equivalent form : $\left(\dfrac{dx}{ds}\right)^2 - \tan^2 \phi = 1$, where $\tan \phi = \dfrac{dy}{dx}$.

We can, therefore, write : $\dfrac{ds}{dx} = \cosh u$, $\tan \phi = \sinh u$, u being some function of x not zero ; it is assumed that $f(x, y)$ is continuous and differentiable over any range considered.

There are two forms of the equation for s, depending on the fact that

$\dfrac{d \tan \phi}{du} = \dfrac{ds}{dx}$. These are :

$$ds = \frac{d \tan \phi}{du} \cdot dx, \quad \text{.................................(1.1)}$$

$$ds = \frac{dx}{du} \cdot d \tan \phi. \quad \text{.................................(1.2)}$$

The simplest solution of (1.1) is given by $\dfrac{d \tan \phi}{du} = \text{constant}$; this represents a set of straight lines, $s = a \,.\, x + b$. The simplest solution of (1.2) is given by $\dfrac{dx}{du} = \text{constant}$; this represents a set of catenaries, $s = a \,.\, \tan \phi + b$.

It would appear that these two forms of linear solution exhaust the "simple" solutions of the fundamental differential equation for the length of a curve referred to rectangular coordinates ; so that the catenary appears as the alternative to the straight line as the "simple" form of a curve.

This conclusion, although possibly surprising, finds a physical expression in the form of a hanging chain, subject to no forces other than its own weight. When it hangs from one extremity alone (uniterminal suspension) it has the form of a straight line; when it is suspended from both extremities (biterminal suspension) it has the form of the "mathematical" catenary, as is well known. The particular catenary curve that it assumes is a function of x only, that is, of the distance between its points of suspension. In each of these two cases the chain is under the action of the minimal possible force, its own weight ; there are no extraneous forces acting upon it. To make it take and keep any other form extraneous forces must be applied and maintained. Thus the straight line and the catenary are both *curves of minimal extraneous force*. They are two instances of a law that the chain takes the simplest form consonant with its mode of suspension and with the external forces acting upon it. Similar reasoning applies to the catenary form of a section of a stretched "cylindrical" soap film, where the force in question is that of surface tension.

The importance of a mathematical demonstration of the "simplicity" of the catenary curve lies in this, that it is the form of the "ideal" curve in certain anatomical structures, such as the modern dental arcade, where gravitational or surface tension forces do not enter sensibly into the mechanical picture (MacConaill and Scher, "The Ideal Form of the Human Dental Arcade." *The Dental Record*, vol. 69, no. 11 (Nov. 1949), pp. 285-302). The present note has been submitted for publication here for that reason.

There is a curious association between the straight line and the catenary, which can be made the basis of a geometrical treatment of this curve. The equation to the curve, usually written $y = a \cosh (x/a)$ can be also put in the form :

$$Y = a \,.\, (\cosh (x/a) - 1) \quad (Y = y - a) \; ; \quad \text{....................(2.1)}$$

this has the effect of transforming all possible curves to a common base-line, so that all have one and the same apex. The intrinsic equation,

$$s = a \,.\, \tan \phi = a \,.\, \sinh (x/a),$$

is, of course, unaltered by this change of y to Y.

There is *some* line, as yet undefined, which cuts all the catenaries defined by the equation (2.1) above, and has the property that along it x/a is constant. Now since this parameter is constant, its hyperbolic sine and hyperbolic cosine are constant also. We can replace x/a by $\dfrac{x}{Y} \cdot \dfrac{Y}{a}$. Since $\dfrac{Y}{a}$ is

equal to $\cosh(x/a)-1$, it follows that $\dfrac{x}{Y}$ is constant. That is, *the locus, $x/a=$ const., is a straight line.* Where this line cuts successive catenaries $\tan\phi\ (=\sinh(x/a))$ is constant also; that is, *the tangents to successive catenaries cut by the straight-line, $x/a=$ const., are parallel.* By replacing x/a by $\dfrac{x}{s}\cdot\dfrac{s}{a}$ it can likewise be shown that x/s is constant, s being defined as the segment of the catenary between the origin and the point where the line, $x/a=$ const., crosses the curve. The locus-lines of constant x/a naturally all pass through the origin. Thus *successive segments of the locus-lines $x/a=$* const. *are directly proportional to the segments of the successive catenaries*—all being measured from the origin. There is no need to labour the developments of this theorem. It may be mentioned, however, that the "segments" of four successive catenaries will have the same anharmonic ratio as the corresponding segments of the line of constant x/a.

M. A. MacConaill.

2165. *Transformations of matrices with zero trace.*

This note is concerned with the following problem: Given a real square matrix A with zero trace, to find a real orthogonal matrix U such that $U^{-1}AU$ is a matrix with null diagonal (*i.e.* a matrix with zero in each position on the leading diagonal). Since symmetry and skew symmetry are invariant under orthogonal transformations, it is clear that only the symmetric part of A is involved in the determination of U. Thus when A is of order 3 the problem is equivalent to each of the following:

(i) to find three mutually perpendicular generators of a given quadric cone;

(ii) to find a triangle inscribed in a given conic and self-conjugate with respect to the absolute conic; .

(iii) to find three mutually perpendicular planes of pure shear in a medium subject to a given system of homogeneous stress (or strain).

It is clear that a necessary condition in each case is the vanishing of the trace of the associated real symmetric matrix. It will be shown that this is also a sufficient condition, by describing a construction by which one can obtain a real orthogonal matrix that transforms a given real $n\times n$ matrix with zero trace into a matrix with null diagonal. The construction is carried out in at most $n-1$ steps, each of which involves nothing more than the solution of a quadratic equation and the multiplication of two matrices of simple form.

A real orthogonal matrix is not uniquely determined by the requirement that it should transform a given matrix with zero trace into a matrix with null diagonal; it will be shown, in fact, that there are $\frac{1}{2}(n-1)(n-2)$ degrees of freedom in the choice of such an orthogonal matrix. In the construction described first, however, there are no degrees of freedom.

Let $A=(a_{rs})$ be an $n\times n$ matrix having a_{rs} in its r-th row and s-th column, where every a_{rs} is real and $\sum_{r=1}^{n} a_{rr}=0$. If (v_{rs}) is an orthogonal matrix, the transform of A by (v_{rs}) is

$$(v_{rs})^{-1}(a_{rs})(v_{rs})=(b_{rs}),$$

where

$$b_{rs}=\sum_{p=1}^{n}\sum_{q=1}^{n} v_{pr}v_{qs}a_{pq}.$$

If the elements on the leading diagonal of A are not all zero, let i be the first integer such that $a_{ii} \neq 0$. Then since $\sum_{r=1}^{n} a_{rr} = 0$, we can find j such that a_{jj} is opposite in sign to a_{ii}, and $i < j \leqslant n$. Let (v_{rs}) be an $n \times n$ matrix such that $v_{ii} = v_{jj} = \cos\theta$, and $v_{ij} = -v_{ji} = \sin\theta$, with unity in each of the remaining positions on the leading diagonal and zero in all other positions. Then (v_{rs}) is evidently an orthogonal matrix, real if θ is real. Moreover, when A is transformed by (v_{rs}), only those elements in the rows and columns numbered i and j are affected. In particular,

$$b_{ii} = a_{ii}\cos^2\theta + a_{jj}\sin^2\theta + (a_{ij} + a_{ji})\cos\theta\sin\theta,$$

and $b_{rr} = 0$ for $r < i$. If we are to have $b_{ii} = 0$, then

$$a_{jj}\tan^2\theta + (a_{ij} + a_{ji})\tan\theta + a_{ii} = 0,$$

i.e.

$$\tan\theta = \{-a_{ij} - a_{ji} \pm \surd((a_{ij} + a_{ji})^2 - 4a_{ii}a_{jj})\}/2a_{jj}.$$

Since $a_{ii}a_{jj}$ is negative, these values of $\tan\theta$ are real.

Thus we can always choose (v_{rs}) so that (b_{rs}) has one more zero on its leading diagonal than A has. The trace being invariant, the matrix (b_{rs}) resembles A in that $\sum_{r=1}^{n} b_{rr} = 0$, so that if $b_{rr} \neq 0$ for some r, we can apply to (b_{rs}) the process just described for A, using a new transforming matrix (w_{rs}). The resulting matrix is the transform of A by the real orthogonal matrix $(v_{rs})(w_{rs})$, and has two additional zeros on its leading diagonal. Ultimately, by repeated operations of this kind, we can find a matrix C with null diagonal, such that

$$C = U^{-1}AU = U'AU,$$

where the real orthogonal matrix U is the product (in order) of those used in the separate stages of the reduction, and U' is the transverse of U.

This method of reducing A to a matrix with null diagonal is, of course, particularly simple if A is in diagonal form. For example, if λ_1 and λ_2 are opposite in sign, the construction gives

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix} = U \begin{pmatrix} 0 & (-\frac{1}{2}\lambda_1\lambda_2)^{\frac{1}{2}} & (-\frac{1}{2}\lambda_1\lambda_2)^{\frac{1}{2}} \\ (-\frac{1}{2}\lambda_1\lambda_2)^{\frac{1}{2}} & 0 & \lambda_1 + \lambda_2 \\ (-\frac{1}{2}\lambda_1\lambda_2)^{\frac{1}{2}} & \lambda_1 + \lambda_2 & 0 \end{pmatrix} U^{-1},$$

where

$$U = \begin{pmatrix} \alpha & \beta/\surd 2 & \beta/\surd 2 \\ -\beta & \alpha/\surd 2 & \alpha/\surd 2 \\ 0 & -1/\surd 2 & 1/\surd 2 \end{pmatrix},$$

and $\alpha = (1 - \lambda_1/\lambda_2)^{-\frac{1}{2}}$ and $\beta = (1 - \lambda_2/\lambda_1)^{-\frac{1}{2}}$.

The construction can be generalised in various ways. For example, if A is a real $n \times n$ matrix whose trace is $n\chi$, and I is the unit matrix of the same order as A, the matrix $A - \chi I$ has zero trace, so we can find a real orthogonal matrix U such that $U^{-1}(A - \chi I)U$ has a null diagonal. Thus $U^{-1}AU$ has χ in each position on its leading diagonal. With a slight modification, the construction can also be used to transform A into a matrix having on its leading diagonal arbitrary numbers whose sum is the trace of A. In this case, however, the transformation, though orthogonal, may not be real.

We now consider the number of degrees of freedom in the most general real orthogonal transformation of a given $n \times n$ matrix A into a matrix with

null diagonal. Suppose that the first r columns of the transforming matrix are known. These correspond to r mutually perpendicular generators of a cone in n-dimensional space. For the $(r+1)$-th, we may take any of the ∞^{n-r-2} generators of the cone which is the intersection of the given cone with the sub-space orthogonal to the first r generators. The total number of degrees of freedom is therefore

$$\sum_{r=0}^{n-2} (n-r-2) = \tfrac{1}{2}(n-1)(n-2).$$

Since a Hermitian matrix can be transformed by a unitary matrix into a real diagonal form, it is clear that it can also be transformed by a unitary matrix into a real symmetric matrix with null diagonal.

The sum of the squared moduli of the elements of a matrix A is invariant under unitary transformations, since it is the trace of AA'. Hence, in particular, if $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of a Hermitian matrix A, and

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

then if A is transformed into a real matrix with null diagonal by a unitary matrix, the elements of the transform are bounded by the numbers

$$\pm\{\tfrac{1}{6}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\}^{\frac{1}{2}}.$$

J. D. WESTON.

2166. *On the formula of the mean.*

It is not an infrequent rider on the formula of the mean

$$f(a+h) - f(a) = hf'(a+\theta h) \qquad \text{.............................(1)}$$

to ask for a proof that, when θ is independent of both a, h, i.e. is an absolute constant, then the function f must be quadratic or possibly linear. The usual proof relies on partial differentiation in a, h and presupposes f differentiable to the second or even the third order. It is, however, possible to obtain the result without further differentiation, and, in fact, we may replace (1) by the less informative identity

$$f(a+h) - f(a) = h\phi(a+\theta h), \qquad \text{............................(2)}$$

where all that I suppose is that f, ϕ are defined for all real values of the variable and are one-valued : this simplifies the essential argument.

Interchanging a, $a+h$ in (2) gives

$$f(a) - f(a+h) = -h\phi\{a + (1-\theta)h\},$$

so that

$$\phi(a+\theta h) = \phi\{a + (1-\theta)h\}.$$

If $\theta \neq \frac{1}{2}$, we can solve (in a, h)

$$a + \theta h = x, \quad a + (1-\theta)h = y$$

for arbitrary x, y. This gives $\phi(x) = \phi(y)$ for any x, y, and so $\phi(x)$ is a constant (A, say). Then, from (2),

$$f(a+h) - A(a+h) = f(a) - Aa,$$

so that $f(x) - Ax$ is again a constant (B, say). Thus, when $\theta \neq \frac{1}{2}$, $f(x)$ has the linear form $Ax + B$ (and θ is irrelevant).

When $\theta = \frac{1}{2}$, the defining formula is

$$f(a+h) - f(a) = h\phi(a + \tfrac{1}{2}h). \qquad \text{.............................(3)}$$

Giving (a, h) the pairs of values $(a, -h)$, $(a-h, 2h)$ we also have

$$f(a-h)-f(a)=-h\phi(a-\tfrac{1}{2}h), \quad f(a+h)-f(a-h)=2h\phi(a).$$

Eliminating f between these three identities gives

$$\phi(a+\tfrac{1}{2}h)+\phi(a-\tfrac{1}{2}h)=2\phi(a), \qquad \text{.........(4)}$$

and therefore

$$\phi(a+\tfrac{1}{2}h)+\phi(a-\tfrac{1}{2}h)=\phi(a+\tfrac{1}{2}k)+\phi(a-\tfrac{1}{2}k), \qquad \text{.........(5)}$$

since h is absent from the right of (4).

With $a=\tfrac{1}{2}(x+t)$, $h=x+t$, $k=x-t$ we can rewrite (5) as

$$\phi(x+t)-\phi(x)=\phi(t)-\phi(0)\equiv\psi(t), \text{ say.} \qquad \text{.........(6)}$$

Then, from (3),

$$f(x+y)-f(x)-f(y)+f(0)=y\{\phi(x+\tfrac{1}{2}y)-\phi(\tfrac{1}{2}y)\}=y\psi(x).$$

By symmetry in x, y this must also equal $x\psi(y)$, and so

$$\psi(x)/x=\psi(y)/y=\text{a constant } (A, \text{ say}).$$

Thus, from (6),

$$\phi(x+t)-\phi(x)=At,$$

which, as in the first part, gives

$$\phi(x)=Ax+B,$$

with B a second constant. Then, from (3) again,

$$\begin{aligned} f(x)-f(y) &= (x-y)\phi(\tfrac{1}{2}x+\tfrac{1}{2}y) \\ &= (x-y)\{\tfrac{1}{2}A(x+y)+B\}, \end{aligned}$$

i.e. $$f(x)-\tfrac{1}{2}Ax^2-Bx=f(y)-\tfrac{1}{2}Ay^2-By=\text{a constant } (C, \text{ say}).$$

This gives $f(x)$ the quadratic form $\tfrac{1}{2}Ax^2+Bx+C$, and completes the proof.

T. W. CHAUNDY.

2167. *Pan-Magic squares of even order.*

N. and W. J. Chater have proved (*Mathematical Gazette*, XXXIII, No. 304) that the determinant of a pan-magic square of even order is zero. An alternative proof, which yields further information concerning the properties of such squares, is set out below.

A pan-magic square of order $2n$ is of the form

$$\begin{array}{c|c|c} a_1 a_2 \;\ldots\ldots\; a_n & A_1' A_2' \;\ldots\ldots\; A_n' & \lambda_1 \\ b_1 b_2 \;\ldots\ldots\; b_n & B_1' B_2' \;\ldots\ldots\; B_n' & \lambda_2 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ k_1 k_2 \;\ldots\ldots\; k_n & K_1' K_2' \;\ldots\ldots\; K_n' & \lambda_n \\ \hline A_1 A_2 \;\ldots\ldots\; A_n & a_1' a_2' \;\ldots\ldots\; a_n' & -\lambda_1 \\ B_1 B_2 \;\ldots\ldots\; B_n & b_1' b_2' \;\ldots\ldots\; b_n' & -\lambda_2 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots \\ K_1 K_2 \;\ldots\ldots\; K_n & k_1' k_2' \;\ldots\ldots\; k_n' & -\lambda_n \end{array} \qquad \text{.........(i)}$$

dashes denoting complements with respect to S/n, where S is the square constant. Thus $A_1'=(S/n)-A_1$.

We have

$$\begin{aligned} \sum_{i=1}^{n} A_i &= S-\sum_{i=1}^{n} a_i' \\ &= S-\sum_{i=1}^{n}\left(\frac{S}{n}-a_i\right), \end{aligned}$$

$$= \sum_{i=1}^{n} a_i.$$

Similarly $$\sum_{i=1}^{n} B_i = \sum_{i=1}^{n} b_i \text{ etc.} \quad \text{.................................(ii)}$$

We shall show that multipliers $\lambda_1, \lambda_2, \ldots \lambda_n, -\lambda_1, -\lambda_2, \ldots -\lambda_n$ can be found, not all zero, such that if they multiply the rows of the square in order from the top as indicated in the figure and then the rows be added, the result will be a row of zeros, *i.e.* the rows are linearly dependent and we conclude that the determinant of the square vanishes.

We have to find λ_i such that

$$\lambda_1 a_i + \lambda_2 b_i + \ldots + \lambda_n k_i - \lambda_1 A_i - \lambda_2 B_i - \ldots - \lambda_n K_i = 0,$$
$$\lambda_1 A_i' + \lambda_2 B'_i + \ldots + \lambda_n K_i' - \lambda_1 a_i' - \lambda_2 b_i' - \ldots - \lambda_n k_i' = 0. \quad (i = 1, 2, \ldots n.)$$

The second set of equations is identical with the first, and the latter may be written

$$(a_i - A_i)\lambda_1 + (b_i - B_i)\lambda_2 + \ldots + (k_i - K_i)\lambda_n = 0. \quad (i = 1, 2, \ldots n.)$$

This set possesses a non-zero solution in the λ_i if

$$D = \begin{vmatrix} (a_1 - A_1) & (b_1 - B_1) & \ldots\ldots & (k_1 - K_1) \\ (a_2 - A_2) & (b_2 - B_2) & \ldots\ldots & (k_2 - K_2) \\ \ldots & \ldots & \ldots & \ldots \\ (a_n - A_n) & (b_n - B_n) & \ldots\ldots & (k_n - K_n) \end{vmatrix} = 0.$$

But we have already proved at (ii) that

$$\sum_{i=1}^{n} (a_i - A_i) = \sum_{i=1}^{n} (b_i - B_i) = \ldots = \sum_{i=1}^{n} (k_i - K_i) = 0.$$

It now follows that $D = 0$ by addition of rows, and hence the result is proved.

We may also prove a theorem concerning the matrices of pan-magic squares of even order, viz.

Theorem.

The product of any three pan-magic matrices of the same even order and constants S_1, S_2, S_3, is a pan-magic matrix of constant $S_1 S_2 S_3$.

Let (a_{ij}), (b_{ij}), be pan-magic matrices of order $2n$ and constants S_1, S_2 respectively, and let (p_{ij}) be their product.

Then

$$p_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk},$$

and the sum of the elements in the kth column of (p_{ij}) is

$$\sum_{i=1}^{2n} p_{ik} = \sum_{i=1}^{2n} \sum_{j=1}^{2n} a_{ij} b_{jk}$$
$$= \sum_{j=1}^{2n} b_{jk} \sum_{i=1}^{n} a_{ij}$$
$$= S_1 \sum_{j=1}^{2n} b_{jk}$$
$$= S_1 S_2. \quad \text{...(iii)}$$

Similarly we may show that the sum of the elements in any row is also $S_1 S_2$.

Consider now the element of (p_{ij}) which occupies a position complementary

to the element p_{ij} in the sense that a_r' is complementary to a_r in the matrix (i) above. This is the element $p_{n+i,\,n+j}$ if $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$.

We have

$$\begin{aligned}
p_{n+i,\,n+j} &= \sum_{r=1}^{2n} a_{n+i,r}\, b_{r,n+j} \\
&= \sum_{r=1}^{n} a_{n+i,r}\, b_{r,n+j} + \sum_{r=n+1}^{2n} a_{n+i,r}\, b_{r,n+j} \\
&= \sum_{r=1}^{n} \left(\frac{S_1}{n} - a_{i,r+n}\right)\left(\frac{S_2}{n} - b_{r+n,j}\right) \\
&\qquad + \sum_{r=n+1}^{2n} \left(\frac{S_1}{n} - a_{i,r-n}\right)\left(\frac{S_2}{n} - b_{r-n,j}\right) \\
&= \frac{2S_1S_2}{n} - \frac{S_1}{n}\sum_{r=1}^{2n} b_{rj} - \frac{S_2}{n}\sum_{r=1}^{2n} a_{ir} + \sum_{r=1}^{2n} a_{ir}\, b_{rj} \\
&= \sum_{r=1}^{2n} a_{ir}\, b_{rj} \\
&= p_{ij}. \qquad\qquad \text{(iv)}
\end{aligned}$$

Similar results may be proved to hold as between the remaining two complementary quadrants of (p_{ij}). Hence complementary elements of (p_{ij}) are equal.

Now let (c_{ij}) be any pan-magic matrix of order $2n$ and constant S_3. Consider the product

$$(p_{ij}) \,.\, (c_{ij}) = (q_{ij}).$$

We show as at (iii) that each row and column of (q_{ij}) sums to $S_1S_2S_3$.

The element of (q_{ij}) complementary to q_{ij} in the case $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, is $q_{i+n,\,j+n}$, and we have

$$\begin{aligned}
q_{i+n,j+n} &= \sum_{r=1}^{n} p_{i+n,r}\, c_{r,j+n} + \sum_{r=n+1}^{2n} p_{i+n,r}\, c_{r,j+n} \\
&= \sum_{r=1}^{n} p_{i,r+n}\left(\frac{S_3}{n} - c_{r+n,j}\right) + \sum_{r=n+1}^{2n} p_{i,r-n}\left(\frac{S_3}{n} - c_{r-n,j}\right) \\
&= \frac{S_3}{n}\sum_{r=1}^{2n} p_{ir} - \sum_{r=1}^{2n} p_{ir}\, c_{rj} \\
&= \frac{S_1S_2S_3}{n} - q_{ij}.
\end{aligned}$$

A similar result relates q_{ij} in complementary positions in the remaining quadrants.

We have proved therefore that the rows and columns of (q_{ij}) have equal sums $S_1S_2S_3$, and that the sum of any two elements in complementary positions is $S_1S_2S_3/n$. It follows that (q_{ij}) is pan-magic.

D. F. Lawden.

1643. . . . she was only sixteen, and had as yet read nothing but Latin and Greek,—unless we are to count the twelve books of Euclid and Wood's *Algebra*, and sundry smaller exercises of the same description. . . .—A. Trollope, *The Last Chronicle of Barset*, ch. lxiii.

REVIEWS.

From Euclid to Eddington. By SIR EDMUND WHITTAKER. Pp. ix, 212. 15s. 1949. (Cambridge University Press)

This latest book by Sir Edmund Whittaker contains the substance of the Tarner Lectures which he delivered at Cambridge in 1947. In it he sets out to trace in all its aspects the development of natural philosophy from its primitive beginnings in early times to the present day, and so to provide a "history of the evolution of concepts and principles, especially such as have provoked long controversies, in some cases still unsettled".

The author has brilliantly succeeded in a formidable task. Not only does this survey make fascinating reading, but it renders an inestimable service to the cause of mathematical physics. Even the specialist may be so absorbed in his own restricted field of activity as to be inadequately informed about many questions, of which the following may serve as examples: What is the significance of affine geometry in mathematical physics? Where in this subject does three-valued logic find an application? What are the distinguishing features of mesons, neucleons, positrons and other such particles? What is meant by the "number of particles in the universe?" All such questions find their answer in this volume.

The subject-matter is treated under five distinct heads, and the first part deals with ideas of space, time and movement. The contributions of Plato and his successors to the idea of physical space and its relation with theoretical geometry are critically examined. Though the necessity of distinguishing between molar (classical), microscopic (sub-atomic) and extra-galactic geometry is now fully realised, the metaphysical fantasies of the Kantian school did much to retard the development of non-euclidean geometry in the nineteenth century. The pure mathematician is free to contemplate many different systems of geometry, but, since the notion of distance is essentially a physical one, it is necessary to meet the challenge implied in the following question: "If we draw a triangle one of whose vertices is at the earth, one is somewhere in the Great Nebula of Andromeda, and one is in the spiral nebula in Canes Venatici, will the triangle be euclidean or non-euclidean?" Physical space is a geometrical system and theoretically the answer will depend upon the nature of the relation between the ten mutual distances of any five points in it. In fact, however, as the author sagely remarks, the true nature of physical space can be decided only by less direct methods.

In the sections dealing with time there is a reference to Tolman's suggestion that relativistic thermodynamics can provide a narrow loop-hole of escape from the customary belief that the end of the world is inevitably predetermined by the principle of the *degradation of energy*.

The first part of the book concludes with an account of the Fitzgerald contraction, the Lorentz transformation and the discoveries which led to the Principle of Special Relativity. This principle may be expressed as a *postulate of impotence* in the form: "It is impossible to detect a uniform translatory motion, which is possessed by a system as a whole, by observations of phenomena taking place wholly within the system." There are many similar postulates, and on them conceivably a complete structure of physical theory could be based. Perhaps the most engaging of them is: "It is impossible to tell where one is in the universe and it is impossible to tell the cosmic time." This postulate not only serves as a foundation for the cosmological theory of Professor E. A. Milne, but also links it up with the "Perfect Cosmological Principle" of Bondi and Gold.

The second part of the book introduces the concepts of classical physics—mass, momentum and energy. The foundation of modern mechanics is traced

to a discovery made by William of Occam, a Franciscan friar, in the fourteenth century. In discussing the celebrated *Five Ways*, or proofs of the existence of God, written by St. Thomas Aquinas, he disagreed with the latter's theory of motion, which was based on that of Aristotle, and hit upon the idea of *momentum*, which he conceived as a sort of non-material cargo carried, for example, by an arrow in its flight. In this and other ways he prepared the ground for the work of Galileo and those who followed.

Though the age which began with Newton and ended with the death in 1907 of Lord Kelvin witnessed the almost arrogant triumph of determinism in physics, yet there were many misgivings about the possibility of " action at a distance " and the true nature of gravitation. Much progress was made, however, with the identification of energy in its various kinetic, potential, thermal, electric and magnetic manifestations, while attempts at a comprehensive unification of the subject, generally by aid of some minimum principle such as " Least Action " met with fair success.

During the nineteenth century many solid and liquid " aethers " with diversified properties were suggested, not so much to explain action at a distance as to provide a medium adapted to sustain the more elusive forms of energy such as occur, for example, in Clerk Maxwell's electro-magnetic theory of light. But all such media involved the implication that it is possible to define " absolute velocity in space ", and the introduction of relativity in 1905 put an end to the search for a quasi-material aether.

So far as the problem of unification in physics is concerned the most remarkable and, indeed, spectacular success has been the identification of mass and energy by a relation due essentially to Einstein, namely,

$$\text{mass} = \frac{1}{c^2} \times \text{energy},$$

where c represents the velocity of light. The determination of the mass of a system of particles has thus become a somewhat complex problem, and G. L. Clark has recently suggested that it may be preferable to define mass by a relativistic extension of Gauss' well-known theorem relating to the flux of gravitational force through a simple closed surface.

A discussion of rotation, angular momentum and some of the complexities which occur in the relativity theory of spin bring this part of the book to a close.

The third part is devoted to the concepts of general relativity. This theory, published in 1915, found its inspiration in the long-cherished ambition to explain gravitation, link it up with electrical and other phenomena and so construct a *unitary theory* of the external world. In the last decade of the nineteenth century Fitzgerald had surmised that " Gravity is probably due to a change in the structure of the aether produced by the presence of matter ". If we replace the word " aether " by " space-time " and " structure " by " curvature ", we arrive at the central proposition of the theory. Its physical basis is subjected to a critical examination in this section. By imposing a suitable " metric " upon space-time it becomes possible to construct at every point (t, x, y, z) a set of accelerated axes corresponding to any prescribed gravitational field. All this constituted a profound revolution in scientific thought. Einstein's ideas, like Faraday's, " may be regarded as reversions to the Cartesian doctrine of space as a *plenum*, in contrast to the principle of action-at-a-distance in a vacuum, which had been generally accepted by the successors of Newton ". " From time immemorial the physicist and the pure mathematician had worked on a certain agreement as to the shares which they were respectively to take in the study of nature. The mathematician

was to come first and analyse the properties of space and time, building up the primary sciences of geometry and kinematics ; then, when the stage had thus been prepared, the physicist was to come along with the *dramatis personae* —material bodies, magnets, electric charges, light and so forth—and the play was to begin. But in Einstein's revolutionary conception the characters created the stage as they walked about on it ; geometry was no longer antecedent to physics, but indissolubly fused with it into a single discipline." Having discussed the more significant consequences of the new theory Professor Whittaker proceeds to deal with the possibility of a unified theory of gravitation and electromagnetism. The concept of affine geometry has assisted in solving certain fundamental problems relating to parallel transport, and it is perfectly feasible so to develop the geometry of space-time in accordance with recognised principles that it will serve to represent more than one branch of physics in a single construction. But a genuine *physical* unification of the forces of nature, such as Faraday imagined, has not been achieved as yet.

The fourth chapter is concerned with the concepts of quantum mechanics. Beginning with the model of the atom introduced by Bohr and Rutherford, it goes on to explain the significance of h, the celebrated action constant of Max Planck, and to deal with the vexed question of the dual behaviour of light and of electrons, both of which must be regarded now as particles and again as waves, if all the observed phenomena are to receive a satisfactory explanation. As in relativity, so in this theory classical modes of analysis in terms of space and time have to be abandoned. Even the notion of a particle must yield to a new and more fundamental element in the description of the external world called a *state*. A " state " is an *ultimate* event which extends over more than one point of space and more than one instant of time. With reference to the *uncertainty principle* of Heisenberg it is pointed out that the models of classical physics do not bear a strict resemblance to reality and, for a system with n degrees of freedom requiring $2n$ data (coordinates and momenta) for its complete specification, the numerical values of only half the data represent all the information that could be possessed even by an omniscient super-being.

The appropriate mathematical technique for dealing with this novel situation is provided by a *wave-function* involving the idea of probability, and even in arguing about simple systems we must be prepared to discard familiar forms of reasoning and have recourse to a " three-valued logic ". " The regularities on which the science of atomic physics is based are statistical regularities, and do not involve complete determinism."

From the purely mathematical point of view one of the most interesting features of quantum physics is to be found in its connection with non-commutative algebra in general and with Cayley's notion of a matrix in particular. It is, moreover, gratifying to find that the classical equations of Hamilton, namely,

$$\frac{dq}{dt}=\frac{\partial H}{\partial p}, \quad \frac{dp}{dt}=-\frac{\partial H}{\partial q}$$

are still valid in quantum theory, provided that p, q, H are suitably interpreted as operators for which the ordinary commutative law is to be replaced by the relation

$$2\pi(pq-qp)=-ih, \quad (i^2=-1).$$

There is in this chapter an interesting account of the discovery of *electron-spin*—a property which, though mystifying from the statistical point of view,

bears some analogy to the polarisation of light waves and satisfies the relativity principle of invariance. The concluding sections deal with the *exclusion principle* and *exchange interaction*.

The fifth and last part of the book is concerned with the researches of the late Sir Arthur Eddington. Though controversial and to be accepted no doubt with some reserve, his conclusions represent a truly amazing adventure in the realm of thought. From the qualitative aspects of the most advanced physical theories Eddington proceeded to extract the most outstanding quantitative consequences. By identifying the rate of expansion of the universe, as measured in connection with the well-known work on the spiral nebulae carried out by American astronomers early in this century, with the theoretical conditions for stability of the Einstein world he derived some very remarkable results. In particular he obtained numerical values for the radius R_0 and the total mass M of this universe.

From these and the other constants arising in different branches of the subject it is possible, by combining them so as to eliminate their dependence upon arbitrary units of mass, length and time, to derive a set of " pure numbers ". When arranged in order of magnitude these are found to be, not more or less uniformly distributed over the number scale, but crowded into three compact groups. Members of the first group are less than 1,900, and include the ratio of the masses of the proton and electron ; those of the second and third groups are of the orders 10^{39} and 10^{79} respectively. Thus the *cosmical number* N obtained when $2M$ is divided by the mass of the hydrogen atom, was found to lie in the third group and to have the value (as calculated from early data)

$$N = 1{\cdot}4 \times 10^{79}.$$

Later he identified it with the number of independent quadruple wave-functions in space of constant positive curvature, namely,

$$N = \tfrac{3}{2} \times 136 \times 2^{256} = 2{\cdot}36216 \ldots \times 10^{79}.$$

This has been called " the number of particles in the universe ", and Eddington believed that it had a fundamental significance in the *structure* of the cosmos both as regards its molar and its microscopic properties. He succeeded, indeed, in establishing relations between N, R_0 and certain atomic constants so as to arrive at the valuation

$$N = 1{\cdot}29 \times 10^{79}$$

in good agreement with the previous value.

Making use of the quantum mechanical notions of *interchange energy* and the *exclusion principle*, he was able to identify the origin of proper mass and predict from purely theoretical considerations the masses of the electron and proton. He went on to show that " both gravitation and exclusion are aspects of the same ultimate principle that is manifested in interchange energy ".

Along such lines as these did Eddington seek to lay the foundations of a unified doctrine of nature. He seemed to be actuated by the belief that all the different kinds of elementary particles would ultimately be recognised as " disguised manifestations of one ultimate particle ".

Throughout this book at every stage Professor Whittaker has sternly resisted the temptation (which must have been strong at times) to enter into mathematical elaboration. The work is a masterpiece of its kind, and the superb quality of the writing is matched only by the skill and competence with which the complexities of the subject-matter are handled. T. A. B.

The Origins of Modern Science, 1300-1800. By H. Butterfield. Pp. x, 217. 10s. 6d. 1949. (Bell)

This is an important and timely work. Apart from the general value of a good book on the history of modern scientific thought, there is a special value at this time, because of the possibility that the history of science may ultimately be one subject in the new general certificate of education. Whatever may be said for and against this proposal, it would clearly be a catastrophe if it were allowed to be seized on as a soft option, concerned with gossip about Cardan's bad temper, Kepler's matrimonial speculations, or the morality of Newton's niece. To be of any value, it must be a study of the development of ideas ; the necessary link, at the school stage, with the general historical background must, of course, be biographical, but the ideas, not the biography, must be the main theme.

During the past hundred years, historians have ceased to confine their attention to political and military history, and have even gone beyond the timid chapter entitled " Literature of the Period " which used to end the volume. Social and economic history have been developed, the influence of the background of ideas on events has been recognised—an outstanding example is Gardiner's classic treatise on early seventeenth-century England —and recently some respect has been shown, for instance by Halévy and Ensor, for the tremendous impact of technical progress on social history. Even so, it is somewhat of a novelty to find a distinguished historian, renowned for his scholarly studies of the political history of eighteenth-century England, devoting a course of lectures to the history of scientific ideas. We are deeply grateful to Professor Butterfield for having undertaken this task, and for having carried it out in so admirable a manner.

The book is so crammed with ideas, and is so provocative of fresh thought, that a brief review is almost certain to be misleading, since only one or two points can be mentioned. The seventeenth century is the century of genius, and should be recognised as a great revolutionary era. The influence and importance of that scientific revolution far transcends that of the Renaissance or the Reformation. The Renaissance was, or was intended to be, a return, a recovery of the all-embracing knowledge which the Greeks had possessed, the Reformation was, or was intended to be, a return, a recovery of the purer practices of the primitive Church. The Galilean-Newtonian revolution was, almost professedly, a deliberate search for new knowledge, new modes o thought, new doctrines of science. This thesis is set forth brilliantly by Professor Butterfield, with clarity, apt illustration, and, perhaps most praiseworthy of all, with a careful avoidance of crude judgments and of epigrammatic antitheses. He is too fine a historian to bow to the school of divine accident, or to the school of economic inevitability. Galileo does not suddenly and for no apparent reason decide to controvert Aristotle by dropping weights from the leaning tower of Pisa ; * Pascal does not devise probability laws because of an economic decree that sea-going commerce will require marine insurance and hence probability must be invented. In fact, one of the most useful pieces of work in the book is the careful drawing of a distinction between experiment and the experimental method, illustrated by reference to Galileo's " thought-experiments ", which tell us what Galileo thought would happen if they were performed ; and on the other hand, it is left " a debatable question how far the direction of scientific interest was affected by technical needs ".

Since through this period it was becoming steadily clearer that the

* A page or two is given to showing how hopelessly flimsy is the evidence for Galileo having ever performed this feat.

ultimate language of science must be mathematics, Professor Butterfield has much to say about our subject; he recognises its importance as the language of science, and, for instance, suggests that if Bacon was "in a certain sense inadequate for the age" it was because he lacked the geometer's eye, and again, that the crucial seventeenth-century development grew from Descartes' marriage of algebra with geometry. But we need not be too quick to acclaim progress: the author reminds us that Fontenelle notes that in the 1680's it was possible to make a good profit out of mathematics!

The earlier chapters, on the "impetus" theory of the fourteenth century, of which the historical importance has only recently been recognised, on the essential characteristics of Copernican thinking, and on William Harvey and his precursors, are more fascinating than the post-Newtonian chapters, perhaps because the account of a period of assimilation is likely to be less enthralling than that of a period of creative thought, but space is not available for discussing these sections nor for indulging a reviewer's vanity by commenting on one or two minor points at which, as it seems to me, Professor Butterfield has pushed an inference a little further than is justifiable. I must be content to repeat that this is a brilliant, thought-provoking book, of the utmost value to anyone who wishes to understand the development of the main ideas behind current scientific thought. T. A. A. B.

Makers of Mathematics. By A. Hooper. Pp. ix, 402. 18s. 1949. (Faber and Faber)

The Rev. A. Hooper, sometime Headmaster of a Warwickshire Preparatory School and R.A.F. Education Officer, was invited some six years ago to Williams College, Mass., as Assistant Mathematical Professor. This book is a reprint of the American edition published in 1948, and the English reader will find American spelling, slight traces of the free and easy style of transatlantic writers and such statements as "1 ton = 907·2 Kg." The phrase "... times greater than" instead of "... times as great as" occurs twice.

The author's intention is "to introduce students not only to the fascination of the story of mathematics and mathematicians, but also of the actual mathematical processes", and on the whole he makes a good job of it; he mentions some eighty "makers of mathematics" from Ahmes to Klein, and, writing largely for the inexpert, gives a slight introduction to the main branches of the subject. In the historical sections he gives illuminating biographical details of the principal figures and his appreciations are generally objective, though he allows his enthusiasm to run away with him in a few cases (*e.g.* Archimedes, Pascal); most of the anecdotes related are those usually to be found in short histories, but there are some less well known, sometimes with rather startling interpretations. Fifty-one pages are devoted to Newton and the calculus, and the account of the Newton-Leibniz controversy is well-balanced, but it seems a pity that Newton's Mastership of the Mint should be condemned as a waste of time. The accounts of the circumstances leading to the birth of the various branches of mathematics are in general adequate and interesting. Mathematics appears to have begun with Ahmes, and there is no mention of the Moscow papyrus.

It is in the pursuit of his second aim that the author's success is unequal. He has set himself the very difficult task of giving, within the compass of a few pages, an insight into the meaning and object of the processes of algebra, trigonometry, coordinate geometry and the calculus, and he assumes that many of his readers will have no mathematical equipment beyond a knowledge of the first four rules of arithmetic. This heavy handicap often necessitates a sketchy treatment and the making of dogmatic statements which

may puzzle the layman, *e.g.* " when the reader has read the section on directed numbers he will see that the product of two negative numbers is a positive number ", although in this section multiplication is mentioned only indirectly in a problem which requires for its solution $(-4) \times 4 = -16$. x^0 is mentioned before it is defined, and fractional indices are introduced without comment in the chapter on Napier and logarithms. The layman may find the explanation of indivisibles and the calculus rather difficult, and having been told with some emphasis that in the relation $dy/dx = 12x^2$, dy/dx is not a ratio, he may be somewhat disconcerted to find almost immediately that $dy = 12x^2 . dx$. Nevertheless, the author contrives to be interesting. I enjoyed his chapter on the development of trigonometry, and Fig. 52 (latitude and longitude) is one of the most satisfying " solid " figures I have seen. Latin and Greek derivations abound ; the history of the word " gnomon " is interesting ; the derivation of " sinus " as a mistranslation of " jiva " via " jaib " (= bosom) is common in American books. In the section on number-systems there is the curious statement that M is a variant of ϕ ; surely $\phi = 500$?

There are very few misprints : λειψιζ on p. 82, an account omitted on p. 120, a decimal point omitted on p. 84 and " gnomon " incorrectly printed on p. 158. There are 125 diagrams, the last representing the twelfth roots of unity. In a bibliography of twenty-seven " books for further reading " the interested reader will find some to please him, though the inexpert may be rather stumped by Cantor's *Transfinite Numbers* and Russell and Whitehead's *Principia*, and one wonders why de Madariaga's *Christopher Columbus* has strayed into this list ; in addition four histories and four books " to be found in the Rare-Book department of many large libraries " are recommended. Two indexes—one historical, one mathematical—complete the book, which should be welcomed by school librarians if they can afford the price.

B. A. S.

Molecules in Motion. By T. G. Cowling. Pp. 183. 7s. 6d. 1950. (Hutchinson)

There is an important sense in which conscientious semi-popular scientific writing is more exacting and more difficult than writing for the expert. The expert can refer his fellow-experts to an original paper for a difficult argument, he can appeal to all the tools—however little known—which mathematics commands, and he can skip a lengthy mathematical proof by the challenging phrase " it can easily be shown ". Not so the expert who addresses the wider public. It is expected to start with that he could, if he so chose, address the experts ; hence we trust him when he writes for less discriminating readers. In addition, he must understand mathematics well enough to make the hard road easy, if less rigorous, and he should be able to construct stimulating analogies without misleading the reader. In all this Professor Cowling's book succeeds admirably and must rank with the best semi-popular works.

The book requires, it is true, familiarity with concepts like mass and force and the fundamentals of algebraic manipulation ; but neither integral signs nor processes of differentiation are allowed to disturb our peace of mind as we pass from a simple proof of Boyle's Law (p. 20), Dalton's Law of partial pressures (p. 34), the gas equation and the specific heats (p. 45) on through the rest of the book. The three chapters following this historical introduction deal with transport phenomena—viscosity, heat conduction and diffusion. The discussion of " the outsides of gas molecules " deals mainly with van der Waals' equation, and there is an excellent discussion (p. 43) of why the size of molecules appears to decrease with increasing temperature. The failures and modifications of the equipartition law are discussed under the heading " the insides of gas molecules " (the term " shapes " would seem more appro-

priate). Several proofs of Maxwell's distribution law are explained, and the concluding chapters deal respectively with the atmosphere and electricity in gases. The former contains a thought-provoking discussion of the escape of the atmosphere from the Earth (p. 153).

When dealing with transport phenomena the author treats at some length (p. 61) the analogy between the formulae for the viscosity and the heat conductivity, and one wonders whether perhaps a more general discussion of transport phenomena (such as that in Champion and Davy, *Properties of Matter*) would not assist the reader. Another point concerns references. While it is undoubtedly correct to avoid all references to original papers in more popular books, one cannot help feeling that an occasional reference which fits into the context is helpful. On p. 79, for instance, the author refers to some interesting experiments on the diffusion of gases which were carried out by Waldmann in 1946. A reference would assist the interested reader to find out all available details. A similar argument applies to cases where more complicated calculations have been mentioned or their results used (pp. 51 and 67).

An interesting analogy occurs in connection with equipartition. The simple law is compared (p. 33) with the communism in an (undoubtedly hypothetical !) primitive society, none of whose members cared how much or how little (energy !) they possessed. The worldly possessions of the members (*i.e.* the energies of the molecules) would average out pretty well. But suppose now (p. 113) that, in such a society, all members give or accept only pound notes (quantum energy states E, $2E$, ... of the molecules !). If very little money is about, only few people will have a pound note (few molecules will be excited out of the ground state and equipartition fails), while if plenty of money is about (higher temperatures) equipartition tends to be approached (cf. also p. 140). There is an unusual analogy in the auxiliary chapter on " the laws of large numbers ", in which the question of whether or not the behaviour of men is determined mainly by economic motives is considered (pp. 117 and 126). Maxwell's " demon " is also briefly discussed in this chapter (p. 125).

The book has a reasonable index and few misprints (*e.g.* in Fig. 4 and on p. 155). It is to be thoroughly recommended. P. T. L.

Higher Algebra for the Undergraduate. By Marie J. Weiss. Pp. viii, 165. 30s. 1949. (John Wiley, New York ; Chapman & Hall)

Books on algebra are sufficiently few that a new one can always be welcomed. This book gives an account of groups, rings, integral domains, fields, algebraic fields, matrices, group rings and ideals. The concepts are developed with great care and clarity, so that the reader should find the book no more difficult to read and understand than the more conventional courses on, for instance, theory of equations.

The work covered is significant, and the importance of the algebraic outlook is receiving constantly increasing recognition in the mathematical world. But as a University textbook on algebra the content of the book is a little disappointing. In so many respects it stops short just too soon to reach results of such deep significance as fully to justify the cold abstraction of the algebraic methods. Thus algebraic fields and groups are well covered. Why not go but one step further to expound the Galois theory of equations which solves almost miraculously so many problems on algebraic equations which have intrigued mathematicians for a millennium? The powerful applications of algebraic methods to other branches of mathematics and physics are not indicated. So vital a tool as the matrix is not discussed until the last chapters of the book.

Nevertheless the book is valuable for its extremely careful and gradual development of the algebraic methods. It might usefully enrich a sixth form library, giving insight and new horizons to potential mathematicians.

D. E. LITTLEWOOD.

Differential Equations. By Harry W. Reddick. 2nd edition. Pp. 288. 24s. 1949. (Wiley, U.S.A.; Chapman & Hall)

This is a well-written book of limited scope. The exposition is clear and readable; the problems are plentiful, and include applications to mechanics and electrical theory. Partial D.E.'s are excluded; the types of ordinary D.E. considered extend to the simpler cases of solution in series, but with omissions of a few topics which would normally form part of a standard course. For instance, the criterion of exactness of the first-order equation is not given, the solution being made to depend on the recognition of "integrable combinations". Singular solutions (and so Clairaut's form) are omitted; resonance is briefly mentioned only in the case when the free vibration is undamped.

The method of undetermined coefficients is the one advocated for finding a P.I. of a linear D.E. with constant coefficients. Algebraical manipulation of the inverse operator is not used, but it is shown that some critical cases (when the R.H.S. of the equation is of a form included in the C.F.) may be dealt with neatly by multiplying by an exponential factor and applying the "shift" theorem in reverse.

The author stresses the importance of physical units and numerical evaluation, a feature which should appeal to engineers. In one of the two examples given of equations of chemical kinetics, however, it is not made clear that the units used must be proportional to the molecular weights. C. G. P.

La matematica dell' ingegnere e le sue applicazioni. I. By G. FUBINI and G. ALBENGA. Pp. viii, 498. 4,000 lire. 1949. (Zanichelli, Bologna)

This volume suggests an interesting comparison with the work of similar scope by Duschek, reviewed in the *Gazette*, XXXIV, pp. 145-7. The comparison is not quite an even one, for Duschek's book is concerned almost entirely with the provision of the mathematical tools needed by the present-day physicist or engineer, whereas Fubini and Albenga give both the tools and some of their applications; and, so far, we have only one volume of Duschek from a promised set of four, and only the first of two volumes for the Italian book, so that no full estimate of total content can yet be made. Common to both books is a striving after precision in statement and in argument, and a refusal to believe that any kind of reasoning will do for the applied mathematician as long as it is moderately plausible.

The Italian book is much more geometrical in flavour, as we might expect from that nation of geometers. Indeed, the analytical side seems to me to be, in one or two places, rather old-fashioned; for instance, I can not easily believe that the engineer feels any urgent need for e^x defined as $\lim (1+x/n)^n$; and would hold that he is much more likely to want it as a solution of problems in which rate of growth is proportional to size, or even as the function whose inverse fills a gap in the scale of integrated powers. The standard of precision is variable; thus the authors are among the few who are careful to observe that in the usual argument giving the derivative of $\sin x$ we need an explicit appeal to the continuity of $\cos x$, yet they quote the integral of $1/x$ as $\log_e x + C$ without, as far as I can see, any warning about the incompleteness of this form.

The volume opens with chapters on algebra: polynomials, determinants, linear equations. Then there is an interesting and valuable discussion on the

meaning of units and dimensions. There follows a long set of geometrical chapters, pleasantly written and well illustrated. Fundamental ideas on segments, angles, directed quantities, vectors, ideal points and homologous figures, cartesian axes and the cartesian aspect of vectors, are dealt with in some detail, so that we are then in a position to tackle chapters on Gauss' theory of optical systems, Monge's descriptive geometry (so seldom studied in this country), and the theory of moments, with a good deal on graphical statics and funicular polygons. Since these are again followed by geometrical chapters, on polar coordinates as a means of defining complex numbers, and on the elements of coordinate geometry in the plane, we are half-way through the book before we come to the calculus ; but it would, of course, be possible and probably desirable for a student to read some of the later chapters before this stage is reached. Four chapters deal with the elements of the calculus, differential and integral, including functions of two variables, and then there is a section on physical applications in which we see how the concepts of derivative and integral will apply to simple problems in thermodynamics, electricity, elasticity, and so on. To conclude the first volume, we have three more geometrical chapters. Chapter XVI defines the ellipse by the constant sum of focal distances, works out the main properties of the curve, and emphasises its applications to moments of inertia and to elasticity. Chapter XVII deals with the other conics and with some special curves of importance, such as the cycloidal curves. Finally, there is a brief section on curves and surfaces in space, with references to moments of inertia, and to Mohr's circle diagram, the graphical construction familiar to students of elasticity.

There are no examples for the reader ; the index to the whole book will appear in Volume II. The printing is admirable, in the best Zanichelli style, and so are the diagrams.

The geometrical flavour, slightly unusual in a book of this type, makes the content seem delightfully fresh, and produces a change in emphasis which might well bring new ideas and additional stimulus to the teacher of technical mathematics who feels his teaching in danger of becoming stale. We look forward with pleasure to the appearance of the second volume.

T. A. A. B.

Leçons sur quelques types simples d'équations aux dérivées partielles. Par E. Picard. Rep. Pp. 214. 700 fr. 1950.

Leçons sur quelques équations fonctionnelles. Par E. Picard. Rep. Pp. 184. 400 fr. 1950. Cahiers scientifiques, 1, 3. (Gauthier-Villars)

Picard's great *Traité*, perhaps the most charmingly written of all the French classics on analysis, stopped at Volume III, though the author had at one time contemplated a fourth volume, and much material for it was available in the form of lectures. It was, we believe, M. Gaston Julia who persuaded Picard to publish this material as small volumes in the series of *Cahiers scientifiques*, and we are now happy to welcome photo-reprints of two of these. The first deals with simple linear partial differential equations, such as Laplace's equation in two dimensions, Fourier's equation of heat conduction, the wave equation, in a series of short, loosely-connected sections. The intention is not to probe deeply along a narrow front, but broadly to survey these primary partial differential equations of mathematical physics, to discuss and compare modes of solution, and particularly to stress the relation with integral equations. The second volume deals with matters of considerable interest which, however, are not readily accessible in books on analysis. The first section studies the equation $f(x)+f(y)=f(x+y)$ in the real domain, and is illustrated by a fascinating application to non-euclidean geometry. In

the second section, the field of numbers is complex, and the existence of a functional equation is related to analytic continuation, and to the problem of proving one-valuedness, illustrated by the Gamma function, the elliptic functions and Poincaré's transcendants. In the third section, the main problem is that of the equation $F(z+1)-F(z)=f(z)$, for a prescribed $f(z)$, and its generalisations, with applications, for instance, to the pseudo doubly-periodic functions. The last section discusses Abel's equation $f\{\theta(x)\}=f(x)+1$, and the generalised Laplace equation $\nabla^2 V=k^2 V$, and its connection with Fredholm's equation.

These are delightful little volumes. Of the style, it is enough to quote, with whole-hearted agreement, what Picard himself says in the preface to the first: "Elle garde l'allure d'un enseignement oral, ne prétendant pas à une forme didactique soignée et étant plutôt une conversation sur des sujêts mathématiques réunis par un lien plus ou moins lâche, ce qui lui permet d'être plus vivant."

T. A. A. B.

L'analysis situs et la géométrie algébrique. By S. Lefschetz. 2nd ed. Pp. 154. 650 fr. 1950. (Gauthier-Villars, Paris)

This is a photostatic reproduction of the original edition of 1926. It is still the only book which gives an adequate account of the far-reaching contributions which the author has made to algebraic geometry, and is quite indispensable to the serious student of the subject. The publishers are to be congratulated on putting the book back in circulation: it is doubtless not their fault that they have had to be content with a reprint rather than a revised edition.

D. B. S.

Mécanique Ondulatoire du Photon et Théorie Quantique des Champs. Par L. de Broglie. Pp. vi, 208. 2,500 fr. 1949. (Gauthier-Villars, Paris)

In this book M. de Broglie develops from a new point of view his theory of the photon, which he has described previously in the following publications:

(1) *Actualités scientifiques*, No. XIII (Hermann, Paris, 1934).

(2) *Actualités scientifiques*, No. XX (Hermann, Paris, 1936).

(3) *Une nouvelle théorie de la lumière* (Hermann, Paris, 1940-2, two volumes).

(4) *Théorie générale des particules à spin* (Gauthiers-Villars, Paris, 1943).

The purpose of the book under review is to clarify the theory and to compare it with the orthodox quantum theory of fields.

The logic of the overall plan, the lucidity of the presentation and the care with which the readers' requirements are borne in mind go to make this book a typical product of the best French scientific writing. No undue strain is put on the reader by expecting him to fill in intermediate mathematical steps, the number of misprints is kept small, and the notation is simple and in agreement with standard practice. On the other hand, there are almost no references to recent papers, and only a few to the classical papers on the subject. On this score the present volume differs from Professor Wentzel's book on the same subject (*Quantum Theory of Fields*, Franz Deuticke, 1943, translated into English and published by Interscience Publishers in 1949), in which a determined effort was made to harness recent work, and present the reader with a balanced account of the present state of our knowledge. Within the stated limitations, then, that is, as M. de Broglie's development of quantised field theories, the work fulfills its purpose admirably.

The book is divided into three parts. In the first, the relativistic wave mechanics of an electron, a photon and a general particle of spin 1 are dis-

cussed. The procedure differs from the usual one in that the field strengths are introduced as matrix element densities and not as is usually done, in analogy with quantum mechanics as the operator counterparts of the corresponding classical variables. The relationship between these two methods of developing the subject is rather similar to that which exists between wave mechanics and quantum mechanics. The results obtained in the two cases are substantially the same. The second part deals with the second quantisation of fields, and the third treats the interaction of photons with matter. There is a final chapter on the multiple-time theory of Dirac, Fock and Podolsky.

The theory is based on the type of analysis which can be found in Wentzel's book (§§ 1–3), the energy-momentum tensor of the field being introduced right at the beginning in the definite form appropriate to the vector meson field (Wentzel, equation (12.46)). Maxwell's equations for a vacuum are obtained in the special case in which the proper mass of the particle is negligibly small. In fact, the book can be considered as a presentation of the theory of particles of spin 1. The author deviates, however, from the generally accepted view, in regarding photons, like mesons, as particles of non-zero rest mass. This conjecture is discussed and defended at various points throughout the book. P. T. L.

La Théorie de la Relativité Restreinte. By O. COSTA DE BEAUREGARD. Pp. 173. 800 fr. 1949. (Masson et Cie, Paris)

From a purely mathematical point of view this is an elegant and, in many respects, novel treatise on the applications of the Special Theory of Relativity. The author's principal object is to present this theory in its most general tensorial form. The Minkowskian space-time device is adopted throughout and made the keystone of the treatise. As far as possible, the classical method of considering simultaneous sets of events in Galilean frames is discarded.

The scope and character of the work is evident from the following brief synopsis of some of the contents : (1) an introduction which discusses, somewhat perfunctorily, the concept of relativity and also the use of tensors in pseudo-Euclidean space ; (2) kinematics and optics of Special Relativity ; (3) relativistic electromagnetic theory, including a discussion of the invariance and conservation of electric charge ; (4) a long chapter, covering more than one-third of the book, on relativistic dynamics ; and (5) a concluding chapter on miscellaneous topics, including the non-viscous fluid, the Eisenhart-Synge-Lichnerowicz theory of turbulence and de Broglie's wave-mechanics.

In a brief preface recommending the book, de Broglie points out that the author's technique leads him to consider ideas similar to those which have recently been introduced into the quantum theory of fields by Schwinger, and he considers that for this reason among others the author's technique must command attention. Nevertheless, the reviewer feels that from the physical aspect the treatise, which incidentally is published in the *Collections d'ouvrages mathématiques á l'usage des physiciens*, is somehow unsatisfying. He cannot express his own reaction more forcefully than by quoting and underlining the following criticism by de Broglie :

" Il est permis de penser que, pour l'initiation des étudiants aux théories relativistes et également pour l'emploi pratique par les physiciens, les méthodes si élégantes de M. Costa de Beauregard ne sauraient remplacer complètement la méthode habituelle. Celle-ci a, en effet, l'avantage indéniable de découler directement des considérations originelles d'Einstein, considérations si profondes et si fondamentalles qui se rattachent directement aux données de l'expérience. . . . " G. J. W.

Lattice Theory. By GARRETT BIRKHOFF. 2nd edition. Pp. xiii, 283. \$6. 1948. American Mathematical Society Colloquium Publications, 25. (American Mathematical Society, New York)

A characteristic feature of modern mathematics is the tendency to study simple structures which occur, combined with structures of different sorts, in diverse branches of mathematics. The study of these simple structures, defined by suitable sets of axioms, tends to unify various mathematical theories. Best known of such structures are groups : abstract algebra and topology have developed many others. The book under review studies one of these abstract algebraic structures, lattices, and shows their power in many fields of thought.

Lattices are partially ordered sets such that every two elements have both a least upper bound and a greatest lower bound. They can also be described as algebraic systems with two operations—formation of greatest lower and least upper bound—obeying certain rules. In a particular case these rules go back to Boole, for his algebra of logic is the earliest study of a lattice, the lattice of subsets of a set, partially ordered by inclusion : here the two operations are set intersection and union. More general lattices were studied by Dedekind in his ideal theory : the integers ordered by divisibility are an example of his type. The general theory of lattices was developed mainly in the last thirty years, under various names—Verbände, structures, etc.

Garrett Birkhoff has been one of the main contributors to the subject. This is the second edition of his book, and it has been greatly expanded, and almost completely rewritten. It gives an account of the great development of the subject since 1940, much of it the work of the author and his school. In addition the presentation has been expanded and improved, so that the book is now much more self-contained and readable. A useful addition for the reader is the inclusion of a large number of interesting examples to be worked as exercises, in addition to much illustrative matter in the text. There are also given many unsolved problems. The book ends with a bibliography of the more important works on the subject and an index ; copious references are also given in the text.

The reader of the book will require to know the elementary ideas of abstract algebra as well as whatever specialised knowledge is needed to understand the various applications. A foreword gives some of the less well-known algebraic concepts which are used, and another gives very briefly the fundamental ideas of topology in terms of the closure axioms.

The first chapter discusses partially ordered sets, giving, *inter alia*, an interesting account of a theory of cardinal and ordinal arithmetic developed by the author and his associates. Chapter II defines lattices and begins their algebraic theory. Chapter III deals with chains—totally ordered sets. It contains excellent accounts of the theory of well-ordered sets, of transfinite induction, of the classical arithmetic of totally ordered sets, and of the connections between the axiom of choice, the well-ordering theorem and Zorn's lemma ; as far as I know this is the only place in the literature where this last subject is written out explicitly, though it is often taken for granted.

The following chapters are concerned with a series of types of lattices subjected to progressively more restrictive postulates : the modular law, the distributive law, the existence of complements, etc., culminating in the Boolean lattices. Each chapter gives an account of the logical connections between the postulates, and of the applications of the type of lattice dealt with. Most important of these are the applications to algebra, to which are given a separate chapter in addition to the sections in other chapters. This chapter develops a theory of the congruence relations on abstract algebras of a very general type, and discusses ideals by means of the congruences

which define them. These congruences form a modular lattice, and the Jordan-Hölder theorem is proved in a widely generalised form as a theorem on modular lattices. The theorem of Kurosh and Ore are next discussed. An important idea introduced by the author is that of a subdirect union of algebras: it is evidently a powerful tool, judging from its applications to the representation theory of distributive lattices and Boolean algebras in later chapters. Other sections deal with projective and affine geometries which, considered as they here are almost entirely in terms of the incidence axioms, are characterised by the modular lattices of their linear subspaces. Generalisations give infinite dimensional geometries, studied by the author and his school. There is also a brief account of the very interesting continuous geometries defined by von Neumann, for which the dimensions correspond to all numbers between 0 and 1.

The last six chapters deal with applications of the theory. Chapter XI deals with applications to set theory, giving an account of Stone's applications of Boolean algebra to topology, and of work of Wallman and Kaplansky which show that a compact topological space is characterised by the lattice of its closed sets, on the one hand, and by the lattice of its continuous functions on the other. An account of Carathéodory's measure theory on Boolean algebras follows. The next chapter deals with applications to logic and probability, rather discursively. There follow chapters on lattice ordered semigroups, which are important in ideal theory, lattice ordered groups, and vector lattices. The latter subject is one which is in rapid growth and important in analysis; it has been found fruitful to supplement the abstract concepts of linear space theory by taking into account the partial orderings which exist in all concrete spaces. A useful account of the algebra and topology of these spaces is given.

The final chapter deals with ergodic theory. This is a theory which slipped its moorings some eighteen years ago, and now keeps up only distant contact with its base in Statistical Mechanics. The chapter gives an account of the abstract theory, with indications of the connection with the older theory and with probability.

In the main, the style of the book is clear and brisk, but on occasion it tends to undue compression, particularly in the illustrative examples. Occasionally this involves actual omissions, as in the statement of the Generalised Induction Principle on p. 38, and in some proofs; and also the enunciation of Theorem 12 of p. 45, in which the condition (HB) should not refer to the sub-basis B as a whole, but to any arbitrary subset of B. The reader must be prepared to rectify these gaps; they will present little difficulty, as will the majority of the textual errors. More annoying are the frequent errors in references to pages in the book. The following mistakes are perhaps worth mentioning. On p. 66, in the penultimate inequality on line 7, $x \smile z$ should be z; and on the next line $x \smile z$ should be $x \smile y$. The statement on p. 246, that the Dualraum of Köthe and Toeplitz coincides with the conjugate space of a vector lattice, when the underlying spaces coincide, seems incorrect, as the example of the space of convergent sequences shows. In fact, the proof in the footnote on p. 246 is invalid in that the element of the Dualraum there constructed may be zero for a non-zero bounded functional.

The study of the simple abstract structures is certain to become an important part of mathematical education. It is illuminating to see which of the simple structures is involved in a given piece of reasoning about a composite structure. It is important, for example, to realise that the Cantor definition of the real numbers depends on the topological structure of the rationals, while the Dedekind definition depends on their order structure as an incomplete lattice; or to understand that the thermometric temperature depends

only on the order structure of the real numbers, while the absolute temperature depends on the algebraic structure as well.

This book is the best introduction in our language to this way of thought. For this reason, and also for the width and wealth of mathematical culture it displays, the insights it gives into a variety of subjects, it will be of interest to all mathematicians. To students of abstract algebra it will be invaluable.

J. L. B. C.

Non-Linear Problems in Mechanics of Continua. Proceedings of a Symposia in Applied Mathematics in 1949. Pp. vii, 219. \$5.25. 1949. (American Mathematical Society, New York)

In this volume are collected papers which were presented during the first Symposium on Applied Mathematics arranged by the American Mathematical Society. The Symposium resulted from the *Report of the Special Committee on Applied Mathematics* which was appointed to investigate the part played by Applied Mathematics in the activities of the Society. Further Symposia are planned, each focusing attention on a fairly restricted field of Applied Mathematics.

The choice of *Non-Linear Problems in Mechanics of Continua* for the first Symposium was a particularly happy one, and all workers interested in such problems will welcome the publication of these papers in one volume. The volume is divided into two main sections, one Hydrodynamics and the other Elasticity and Plasticity, and well-known writers in these fields have contributed. The first section includes two papers on incompressible flow of a fluid with free boundaries, one being concerned with the problem of the wake and the other with the breaking of waves in shallow water. Work on boundary layer and turbulence is also included, but the majority of the papers deal with the flow of compressible fluids. Two-dimensional flows, the method of characteristics in the three-dimensional case, and shock wave problems, are considered.

In the second half of the volume, papers deal with large deflection theory of plates, and problems are solved for the special cases of rectangular and circular plates. The edge effect in bending and buckling is also discussed. There is one paper on general elasticity theory of finite displacements, and short contributions on plasticity are also included.

Further volumes on other topics in Applied Mathematics will be awaited with interest.

A. E. Green.

Statistics—An Intermediate Text-book. I. By N. L. Johnson and H. Tetley. Pp. xii, 294. 20s. 1949. (Published for the Institute of Actuaries and the Faculty of Actuaries by the Cambridge University Press)

Until recently, comparatively little attention was given to general statistics in the examinations of the Institute of Actuaries. The amount of reading required for the first examination was small and was confined to a few chapters in the official mathematical textbook ; and the questions set for solution in the examination papers were to a large extent simple arithmetical examples on statistical measurements. The framework of the syllabus has now been entirely altered and a more formal study of statistics is required for Part I of the examinations. To quote from a booklet issued in 1946 by the Institute of Actuaries, the new syllabus is " designed to secure increased attention to the forging of links between statistical theory and actuarial subjects ".

The book under review has been prepared mainly to satisfy the needs of actuarial students, and the first volume is intended to cover the statistics for Part I of the examinations. A comparison between the syllabus and the contents of the book shows that the authors have omitted nothing that the

examinee might desire in his reading. In fact, at first sight, one may be easily led to the conclusion that if the other subjects that the Part I student has to read—finite differences and compound interest—cover as much ground as is covered by this textbook, the average examination candidate will need to study hard in order to be sure of passing the examination within a reasonable time.

In their introduction the authors state that the contents of the book are divided roughly into three sections—descriptive statistics, probability and statistical hypotheses and tests. The chapters devoted to descriptive statistics are interesting and are, on the whole, well-written, although there are some points to which criticism may be directed. For example, in paragraph 2.5, the shape of the histogram in Figure 2.2 is stated to be with one small exception typical of the single-humped curve ; in paragraph 3.8 the reason given for the fact that, in the same example, the median does not lie between the mean and the mode is that the distribution does not approximate very closely to that type of curve. Again, in paragraph 3.2, it is said that the arithmetic mean is more difficult to calculate than some others of the central measures. No evidence is given that this is so, and, in fact, from what follows in this chapter the student might be likely to doubt the accuracy of the statement.

In the introduction to probability it is noteworthy that the authors have adopted the formal hypothesis, based on experimental evidence, that for a given value of N, positive numbers p, α and β exist such that K per cent of the observed values of the relative frequency (n/N, where n is the number of successes out of N trials) lie between the limits $p+\alpha$ and $p-\beta$. As N increases without limit α and β tend to zero, p remaining constant at a value between 0 and 1. Following J. E. Kerrich, K is given the quite arbitrary value of 99 per cent. (It may here be remarked that the actuarial student will be well advised to beg, buy or borrow a copy of Kerrich's book, *An Introduction to the Theory of Probability*, a study of which will help in appreciating the experimental approach to the conception of the subject.)

The chapter on Probability (Fundamental) is straightforward and is taken at leisurely speed : it should cause the student little difficulty, With the next chapter, however, the pace quickens. The reader is hurried along through the properties of random variables to a not very satisfactory paragraph on Tchebyshev's inequalities, the proof of which is given for a continuous random variable only. It is at least doubtful whether the average Part I student could prove the theorem for other than a continuous variable—or, indeed, whether he would attempt to prove it. The remark at the end of the first sub-paragraph in paragraph 7.15, " the discontinuous case being left to the reader ", is strangely reminiscent of that of the writers of mathematical textbooks forty and more years ago, where the difficult proofs were often generously left " as an exercise to the student ". In fairness to the authors, however, it must be said that the discussions in this and the next two chapters on probability are clear and helpful, although it is likely that the reader with limited mathematical attainments will find the going hard.

The chapter on simple statistical tests deals mainly with population means, and to that extent is useful to the actuarial student. The paragraphs on quality control are scrappy, and the need for their insertion in this book is not evident.

Taken as a whole, the textbook is an advance on other books on intermediate statistics. As has been said above, it covers admirably the requirements of the new syllabus for Part I of the Institute of Actuaries examinations—and, in the circumstances, one cannot ask for more.

Finally, the statement in the Preface that, in a book such as this, " there must inevitably be a number of errors," is rather naive. Are errors misprints

or do they include such mis-statements as " elementary school children " (page 3) and " this data " (page 40)? The term " elementary school children " has no meaning now—such children attend primary schools—and " data " should not be treated as singular in one place and, correctly, in others, as plural (cf. page 91). The apologia was unnecessary : the authors need have no fear that errors, misprints, etc., will not be pointed out to them by friends, readers and reviewers. H. F.

Transformation Calculus and Electrical Transients. By S. GOLDMAN. Pp. xiv, 439. \$8.35. 1949. (Prentice-Hall, New York)

The modern methods of finding the transients in electrical transmission lines make a demand on the electrical engineering graduate for further reading in mathematics, especially in the theory of functions of a complex variable, Laplace transforms, Fourier integrals, partial differential equations and the study of special functions, such as Bessel functions. In this book, which forms a suitable course of reading for a post-graduate student or for the better undergraduate during his last year, all the above-mentioned topics are dealt with, but with special reference to their connection with Laplace transforms. There are in addition two introductory chapters on determinants and general circuit theory. In every chapter there are very well-chosen electrical examples, and in spite of the fact that mathematics is very closely linked with its applications throughout, the difficult mathematical points are not shirked ; where there is not room to explain a point of rigour in detail ample references are made to standard mathematical texts.

In Laplace transforms s is used as the variable instead of p, which is reserved to mean d/dt ; moreover, the Laplace transform is defined so that the Laplace transform of $t^n/n!$ is $s^{-(n-1)}$. Although the latter is used by pure mathematicians more than s^{-n}, it does seem much easier for the engineer or physicist to remember when the Laplace transform of 1 is taken to be 1. On p. 229 it is stated that under certain conditions p and s are equivalent ; this is so only in the sense that $p(t^n/n!)=s\{s^{-(n-1)}\}$. In any case, seeing that p is used in many books as the variable in the Laplace transform, it would be less confusing to keep to D for d/dt and D^{-1} for the indefinite integral.

Apart from this question of nomenclature the book can be thoroughly recommended, not only for its chapters on Laplace transforms, but also for its other chapters, which any engineer will find to be very useful introductions to the topics mentioned at the beginning of this review. H. V. LOWRY.

Ludwig Schläfli (1814-1895). Gesammelte Mathematische Abhandlungen. I. Pp. 392. Sw. fr. 54. 1950. (Birkhäuser, Basel)

This is the first of three volumes in which the collected papers of Schläfli are being published. Today Schläfli is chiefly remembered for two things, his work on the twenty-seven lines on the general cubic surface (which will appear in the second volume), and his posthumous work *Theorie der vielfachen Kontinuität*. This enormous work was completed in 1852, but owing to its great length was not published until 1901. It occupies the last 220 pages of the volume under review, and completely overshadows the shorter papers, on a variety of topics in geometry and analysis, which complete the volume.

The *Theorie* is a document of great interest, and it is interesting (but perhaps unprofitable) to speculate on the course along which geometry might have developed in the latter part of the nineteenth century had it been published at the time it was written. It is the earliest discussion, on a serious scale, of geometry in space of more than three dimensions. The space which is envisaged is euclidean, and the whole work is essentially an attempt to extend to space of any number of dimensions the metrical properties of

linear spaces and quadrics. As C. Segre observes in his article in the *Encyclopädie*, the actual development of higher geometry was on projective lines, and the abstracts of Schläfli's work which were published at the time attracted scant attention. By 1901 the point of view of the *Theorie* can hardly have been congenial to a generation of geometers revelling in the beauties which the early Italian projective school had laid open to view.

Nevertheless, Schläfli's contribution is of more than merely historical interest, and it reveals great insight and analytical power. It is divided into three parts. The first deals with linear spaces and the mensuration of figures bounded by them. This part concludes with the discussion of the regular polytopes in higher space, and the results have been made familiar in Coxeter's recent book. The second part deals with the geometry of the hypersphere. It centres round the deduction and application of an expression for the content of a spherical simplex (Plagioschem), and particular reference is made to simplexes which arise by subdivision from the central projections of regular polytopes on to their circumspheres. This discussion involves heavy analysis, but leads to interesting results. The last division of the paper is devoted, in the main, to generalisations of the diametral properties of central quadrics, confocals, orthogonal systems and the like, and discusses various questions of curvature which belong to the province of differential geometry.

It remains to add that the book is beautifully printed, and is produced in a way worthy of the occasion which called it forth. J. A. T.

Premiers principes de géométrie moderne. Par Ernest Duporcq. Troisième edition par Raoul Bricard. Pp. 174. 375 fr. 1949. (Gauthier-Villars)

The reappearance of "Duporcq" will give considerable satisfaction. Here is all the clarity which attracted so many readers when it was easily available. Little need be said about such a standard work, except to recommend those who have still to meet it not to delay any longer.

It is perhaps worth while to add that the book affords an excellent introduction to the reading of mathematics in a foreign language, so that two sound purposes are covered by a single purchase.

E. A. Maxwell.

Warne's Metric Conversion Tables. Designed by O. Klein and computed by Scientific Computing Service, Ltd. Pp. 104. 15s. 1950. (Warne)

This volume provides a rapid means of conversion from Imperial or American measures into metric measures, and *vice versa*. There are about 80 tables, for weight, length, speed, surface, volume, capacity, pressure. Dr. J. C. P. Miller contributes a useful foreword on basic factors of conversion and on accuracy, T. A. A. B.

1644. I suppose there is a similar "kink" in my brain which makes me unable to remember algebraic formulae, telephone numbers, or motor-car numbers, or any meaningless conjunction of figures, though I can remember the list of Byzantine Emperors or the Departments of France easily enough, because they have some meaning.—Sir Charles Oman, *Memories of Victorian Oxford*, (1941), p. 54.

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